Unit 2

First-order differential equations

Introduction

This unit introduces differential equations. This topic is of central importance in physics because many physical laws are expressed as differential equations. Differential equations also appear in many other areas of science and applied mathematics. This introduction gives a brief outline of the topic.

We begin by discussing the solution and derivation of a simple differential equation, namely

$$\frac{df}{dx} = -A\,f(x), \tag{1}$$

where $A > 0$ is a constant. This is called a *differential equation* because it contains a *derivative* of an initially unknown function $f(x)$. It is solved by finding a function $f(x)$ that satisfies the equation. You will see how it is solved, and how it arises in two different physical situations. This illustrates several important points about differential equations in quick succession before the body of the unit discusses them at a slower pace.

In some cases differential equations can be solved by guessing the solution. Noting that equation (1) is very similar to equation (19) of Unit 1, we try the function $f(x) = \exp(-Ax)$. The derivative of this function is

Equation (19) of Unit 1 is
$$\frac{d\exp(x)}{dx} = \exp(x).$$

$$\frac{df}{dx} = -A\exp(-Ax).$$

However, we assumed that $\exp(-Ax) = f(x)$, so this shows that our guess satisfies

$$\frac{df}{dx} = -A\,f(x),$$

which is the differential equation that we wished to solve.

The same reasoning works if this **trial solution** is multiplied by any constant: if $f(x) = B\exp(-Ax)$, then $df/dx = -AB\exp(-Ax) = -A\,f(x)$. So we see that

$$f(x) = B\exp(-Ax) \tag{2}$$

is a solution of the differential equation (1) for any value of B. The reason why the trial solution works in this case is easy to understand. The exponential function $\exp(x)$ has the property that when you differentiate it, you get the same function back again. The differential equation (1) describes a function with almost that property, so it is natural to seek a solution of this equation in the form of equation (2).

You have now seen an example of a differential equation and its solution. But how does this differential equation arise in applications? We will discuss two examples.

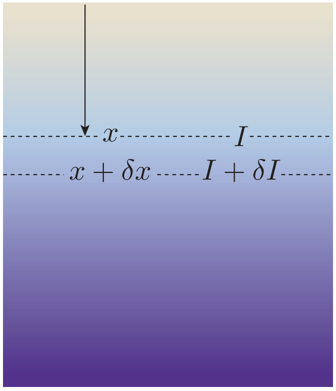


Figure 1 Intensity of light I in the sea at depth x

First, we consider a model for light penetrating into the ocean depths. Sunlight is absorbed in seawater, and almost no light reaches the deepest parts of the oceans. If you were involved with underwater engineering or investigating the behaviour of plankton, which can adjust their depth depending on the light intensity, you might wish to understand how the light intensity decreases as the depth increases. Let the intensity of light be $I(x)$ at depth x below the surface (Figure 1).

We would like to be able to find the function $I(x)$. This is done by finding a differential equation satisfied by $I(x)$. The differential equation is found using the following argument.

Consider what happens as we go from a depth x, passing through a thin layer of seawater of thickness δx to a slightly larger depth $x + \delta x$. (It is conventional to write the small change in any quantity X as δX, where δ is the Greek letter 'delta'). On passing through this thin layer, the intensity of light changes by an amount δI, due to absorption of light. This change is negative because the light intensity is reduced. We might reasonably expect that the amount of light absorbed is proportional to the amount of light entering the layer. Also, the amount of light absorbed by a thin layer is expected to be proportional to the thickness of the layer, so

$$\delta I = -AI\,\delta x,$$

where A is a constant (called the **absorption coefficient**) that describes how effectively seawater absorbs light.

Now divide both sides of this equation by δx, to obtain

$$\frac{\delta I}{\delta x} = -AI.$$

It is assumed that δx is small. If we take the limit as δx tends to zero, we get a differential equation for $I(x)$:

$$\frac{dI}{dx} = -A\,I(x). \tag{3}$$

This is the same as the equation we have already solved, apart from a change in the name of the function from $f(x)$ to $I(x)$. The solution is therefore

$$I(x) = B\exp(-Ax),$$

for some constant B. This solution is plotted in Figure 2 for $A = B = 1$. It has the expected property that the intensity I decreases as the depth x increases. The decrease is rapid in the upper layers but more gradual in the murky lower depths. A function of this type is said to exhibit **exponential decay**.

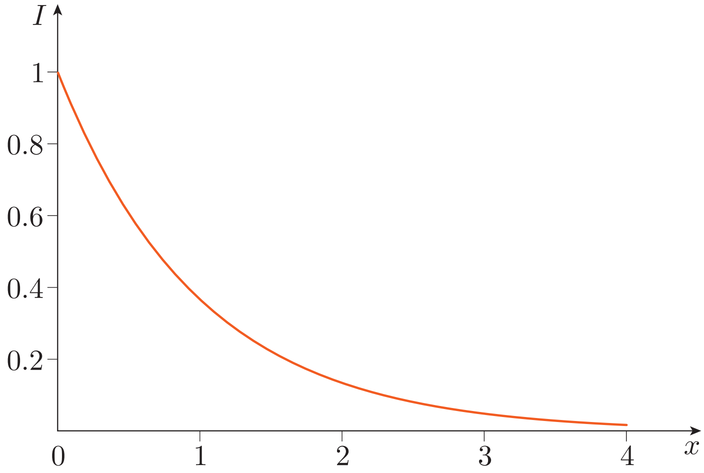


Figure 2 The graph of $I(x) = B\exp(-Ax)$ for $A = B = 1$

Notice that a complicated situation has been made much simpler by considering a very thin layer, in which the absorption of light can be taken to be proportional to the thickness of the layer. This approximation is valid for a thin layer but not for a thick one (as shown by the non-linear shape of the graph in Figure 2). The beauty of this approach emerges when we imagine the layer becoming infinitesimally thin. In this limit, the approximation becomes exact and, at the same time, we obtain an equation involving derivatives, that is, a *differential equation*. So here is mathematical alchemy – we start with an approximation that is reasonable only on a small scale, and use it to obtain a differential equation that avoids this approximation and has solutions that apply on a large scale.

Into the deep

In the ocean, the light that is absorbed least and penetrates furthest beneath the surface is blue in colour. In the clearest ocean water, the absorption coefficient of blue light is $A \simeq 2 \times 10^{-2}\,\mathrm{m}^{-1}$, while that for red light is 100 times greater. At a depth of 200 metres, the intensity of blue light is reduced by a factor $\exp(-4) = 0.018$, while the intensity of red light is reduced by a factor $\exp(-400) = 2 \times 10^{-174}$; essentially no red light is left at this depth. So deep-sea divers see their way around in dim blue light. Most life in the oceans, and all plant life, is concentrated in the top 200 metres of water because deeper waters are too dark to sustain photosynthesis. The ocean floor is generally much further below the surface; it is completely dark, and creatures like the anglerfish (Figure 3) have evolved to make light for themselves via chemical reactions.

Figure 3 An anglerfish creates light to attract its prey

Radioactive decay provides another context in which differential equation (1) appears. In this case, atoms of a radioactive substance ‘decay’; that is, they release radiation and are transformed into other types of atom.

The term 'continuous variable' here means that $N \in \mathbb{R}$. We are also assuming that N is large and *positive*.

At time t, a given piece of radioactive material contains N 'parent' atoms (where N is a very large number). Because the number of atoms is very large, we will treat N as a continuous variable, rather than an integer. The radioactive 'parent' atoms are unstable and can break up to produce a 'daughter' atom and an energetic particle. By considering small intervals of time δt, it can be shown that the differential equation for the number of parent atoms as a function of time is

$$\frac{dN}{dt} = -AN,$$

where A is a constant that depends on the parent atom type. This is the same equation as equation (1), with different names for the symbols, and the solution can be found from equation (2) by changing the names of the symbols. The solution is therefore

$$N(t) = B\exp(-At).$$

Setting $t = 0$, we get $N(0) = B$, so we identify the constant B with the number of atoms present at time $t = 0$, i.e. the initial number of parent atoms at $t = 0$, which we denote by N_0. Hence

$$N(t) = N_0 \exp(-At). \tag{4}$$

The constant A is called the *decay constant*. It is large for types of parent atom that decay rapidly. The decay rate is often expressed in terms of the *half-life* of the atoms, $T_{1/2}$, which is the time taken for half of the parent atoms to decay. To determine the relation between the decay constant and the half-life, set

$$\frac{1}{2} = \frac{N(T_{1/2})}{N(0)} = \exp(-AT_{1/2}).$$

Taking the logarithm of this gives $\ln(1/2) = -AT_{1/2}$, so

$$T_{1/2} = \frac{\ln 2}{A}.$$

Figure 4 Skull of a smilodon

Radiocarbon dating

Atoms of carbon-14 are produced in the upper atmosphere by collisions between cosmic rays and molecules. They are unstable, and decay with a half-life of approximately 5644 years. Carbon-14 is absorbed by an organism when it is alive, and decays after the organism has died. By measuring the ratio of the carbon-14 atoms in a dead organism to those in a living one, the number of years since death can be estimated, using equation (4). For example, Figure 4 shows the skull of a smilodon (a sabre-toothed cat that was larger than a modern tiger). Radiocarbon dating of specimens like this reveals that such cats died about 10 000 years ago.

These examples of light intensity and radioactive decay illustrate some significant points.

- The two examples gave rise to essentially the same differential equation, just with different symbols. You need to be able to recognise when a differential equation is of the same form as one that you have seen before.
- In each case, the differential equation was obtained by considering small changes in physical variables (δx and δt). Similar steps are used in the derivation of many differential equations. However, you will not be asked to derive differential equations in this module; instead, we will focus on ways of solving them.
- The solution was obtained by guessing a suitable function. Judicious guesswork can help us to solve some differential equations, but you will see that systematic methods allow us to solve many more.

Study guide

This unit introduces some basic ideas about differential equations, before considering in detail *first-order differential equations*, so called because they involve only derivatives of first order.

Section 1 gives an example of how a differential equation arises in a mathematical model. It also introduces some basic definitions and terminology associated with differential equations and their solutions.

Sections 2 and 3 develop exact methods for solving first-order differential equations of various special types. From the point of view of later studies, these sections contain the most important material in this unit, which will be built on later in the module.

Some integrals cannot be evaluated by hand. In a similar way, many differential equations cannot be solved exactly. This unit therefore ends by looking at *qualitative methods* in Section 4 and *numerical methods* in Section 5. These sections will not be tested in continuous assessment or in the exam, but it is important that you read them in order to gain an insight into what progress can be made when exact solutions of differential equations lie beyond our grasp.

1 Some basics

The Introduction gave a quick and sketchy overview of the subject of differential equations in order to give you some feeling for the scope and significance of this vast topic. Here we start again from the beginning, developing the subject at a slower pace.

An important class of equations that arises in mathematics consists of those that feature the *rates of change* of one or more variables with respect to one or more others. These rates of change are expressed mathematically by *derivatives*, and the corresponding equations are called *differential equations*. Equations of this type crop up in a wide variety of situations. They are found, for example, in models of physical, astronomical, electronic, economic, demographic and biological phenomena.

First-order differential equations, which are the particular topic of this unit, feature derivatives of order one only; that is, if the rate of change of variable y with respect to variable x is involved, the equations feature dy/dx, but not d^2y/dx^2, d^3y/dx^3 or any higher-order derivatives. In many cases we consider the rate of change of a variable with respect to time, but we also discuss differential equations where the independent variable is position, or occasionally some other variable.

When a differential equation arises, it is usually an important aim to *solve* the equation. For an equation that features the derivative dy/dx, this entails expressing the *dependent variable* y directly in terms of the *independent variable* x. The process of solution requires the effect of the derivative to be 'undone'. The reversal of differentiation is achieved by integration, so it is to be expected that integration will feature prominently in the methods for solving differential equations. The solution can be attempted symbolically, to get an exact formula, or numerically, to get approximate numerical values that can be tabulated or graphed.

Subsection 1.1 describes a situation that leads naturally to a first-order differential equation – in this case, one that is slightly more complex than equation (1).

Subsection 1.2 then introduces some important terminology. It explains what is meant by the term 'solution' in the context of first-order differential equations, and brings out the distinction between the *general solution* and the various possible *particular solutions*. The specification of a constraint (in the form of an *initial condition*) usually allows us to find a unique function that is a particular solution of the differential equation and also satisfies the constraint.

1.1 Where do differential equations come from?

To illustrate how differential equations arise, we consider an example drawn from biology.

Suppose that we are interested in the size of a particular population, and in how it varies over time. The first point to make is that the size of any population is measured in integers (whole numbers), so it is not clear how differentiation will be relevant. Nevertheless, if the population is large, say in the hundreds of thousands, a change of one unit will be relatively very small, and in these circumstances we may choose to model the population size by a *continuous* function of time. This function can be written as $P(t)$, and our task is to show how $P(t)$ may be described by a differential equation.

Let us assume a fixed starting time (which we label $t = 0$). If the population is not constant, then there will be 'joiners' and 'leavers'. For example, in a population of humans in a particular country, the former are those who are born, or who immigrate into the country, while the latter are those who die or emigrate. For our simple model we will ignore immigration and emigration, and concentrate solely on births and deaths.

In the small period between t and $t + \delta t$, it is reasonable, as a first approximation, to expect the number of births to be proportional to the population size $P(t)$ at time t, and to the time interval δt. A similar argument applies to the number of deaths, so we can write

$$\text{number of births} \simeq b\,P(t)\,\delta t,$$
$$\text{number of deaths} \simeq c\,P(t)\,\delta t,$$

where b and c are positive constants known as the *proportionate birth rate* and the *proportionate death rate*, respectively.

The *change* δP in the population over the time interval δt is the number of births minus the number of deaths in that interval. So we have

Births appear with a positive sign because they add to the population; deaths appear with a negative sign because they subtract from the population.

$$\begin{aligned}\delta P &\simeq b\,P(t)\,\delta t - c\,P(t)\,\delta t\\ &= (b-c)\,P(t)\,\delta t.\end{aligned}$$

Dividing through by δt, we obtain

$$\frac{\delta P}{\delta t} \simeq (b-c)\,P(t).$$

The approximations involved in deriving this equation become progressively more accurate for shorter time intervals. So, finally, by letting δt tend to zero, we obtain

This is the step that requires P to be a continuous (rather than discrete) function of t.

$$\frac{dP}{dt} = (b-c)\,P(t). \tag{5}$$

This is a *differential* equation because it describes dP/dt rather than the eventual object of our interest (the function $P(t)$ itself).

We can simplify the above equation by introducing the *proportionate growth rate* r, which is defined to be the difference between the proportionate birth and death rates: $r = b - c$. Then our model becomes

$$\frac{dP}{dt} = rP, \tag{6}$$

where r is a constant (which may be positive, negative or zero).

Equation (6) may look familiar: it is essentially the same as the equation discussed in the Introduction (the only difference being the sign on the right-hand side). So it should not surprise you that the solution is of the form

$$P(t) = P_0 \exp(rt), \tag{7}$$

where P_0 is the initial population at time $t = 0$.

For $r > 0$, this leads to the prediction of an exponential growth in population; for $r < 0$, it predicts an exponential decay in population; and for $r = 0$, the population is predicted to remain constant. These three possibilities are sketched in Figure 5.

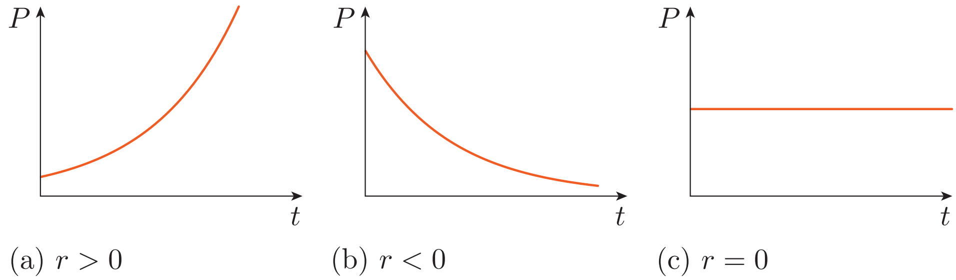


Figure 5 Population models obtained from equation (6), where r is a constant

From equation (6), we see that the proportionate growth rate r is given by

$$r = \frac{1}{P}\frac{dP}{dt},$$

and up to this point, we have assumed this to be constant. However, this is not entirely realistic. For example, if r is a positive constant, then an exponential increase in population may be sustained for a while, but we know that such growth cannot go on forever. When the population is low, we may assume that there is the potential for it to grow (assuming a favourable environment) and the proportionate growth rate r should be high. But when the population becomes too large, competition for basic resources such as food is bound to restrict further growth, and the proportionate growth rate will be lower. We therefore arrive at the conclusion that r will be a function of P, decreasing as P increases. This decline in the proportionate growth rate ensures that unlimited exponential growth will not occur.

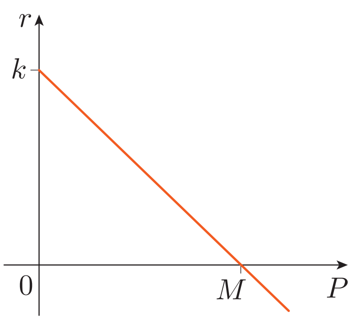


Figure 6 A plot of $r(P)$ as given by equation (8)

A particularly useful model arises from taking $r(P)$ to be a decreasing *linear* function of P, as shown in Figure 6. We write this as

$$r(P) = k\left(1 - \frac{P}{M}\right), \tag{8}$$

where k and M are positive constants. From this formula, you can see that the proportionate growth rate r is modelled as decreasing linearly with P, from the value k (when $P = 0$) to the value 0 (when $P = M$). For $P > M$, the proportionate growth rate is negative.

Using this expression for r, the differential equation satisfied by P becomes

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right). \tag{9}$$

This is well known to biologists as the **logistic equation**. You will see how to solve this equation exactly in Section 2, and how to analyse its qualitative behaviour in Section 3.

It is worth noting that not all differential equations are derived like those above, from first principles using small increments. In science, many differential equations come from physical laws that are framed directly in terms of the rates of change of physical quantities. For example, *Newton's second law* tells us that

$$F = m\,\frac{d^2x}{dt^2},$$

where F is the force acting on a particle of mass m, and d^2x/dt^2 is the acceleration of the particle in the direction of the force. This law produces a large variety of differential equations whose precise form depends on the nature of the force.

The Voyager missions

A spectacular example of the use of Newton's second law is illustrated in Figure 7, which shows the trajectories of the two Voyager spacecraft, launched in 1977.

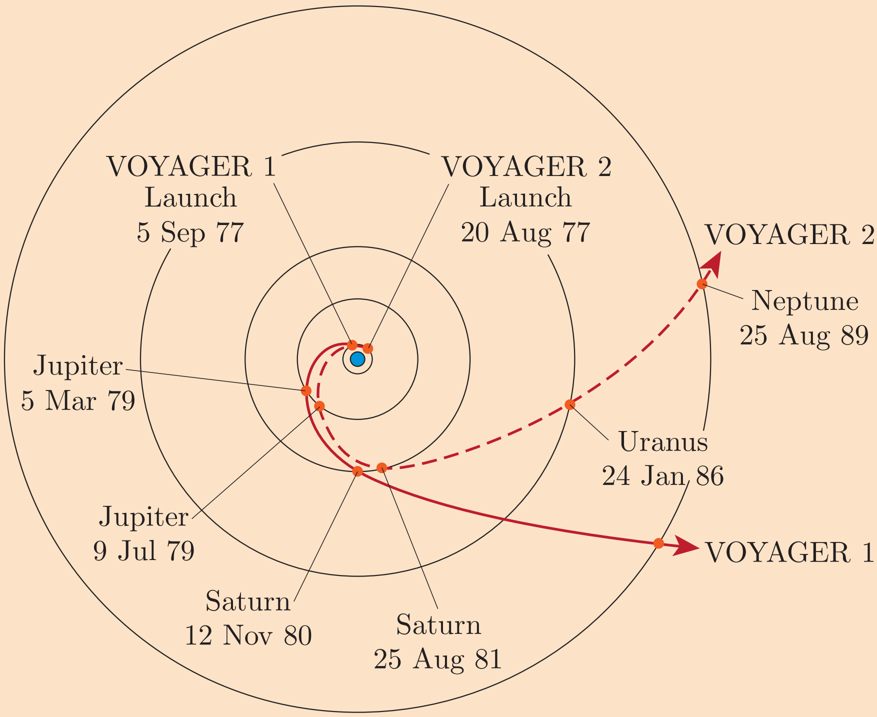


Figure 7 The trajectories of the two Voyager spacecraft were planned by solving differential equations

These trajectories were carefully designed to enable the spacecraft to pass close to various planets and eventually to escape from the Solar System. The trajectories were planned by solving Newton's second law for the position and velocity of each spacecraft. The resulting

Figure 8 A montage of images taken by the two Voyager spacecraft, showing Saturn, its rings and six of its moons

differential equations are complicated by the fact that the forces depend on the positions of the planets, which depend on time. Such equations are much more complicated than the logistic equation, and in practice are solved on computers using numerical methods.

Figure 8 shows a fruit of these labours: close-up views of Saturn and its moons. The moon in the foreground is Dione, which is composed mainly of water ice.

Exercise 1

Suppose that a population obeys the logistic equation (with a proportionate growth rate given by equation (8)), and that you are given the following information. When $P = 10$, the proportionate growth rate is 1, and when $P = 10\,000$, the proportionate growth rate is 0. Find the corresponding values of k and M.

1.2 Differential equations and their solutions

This subsection introduces some of the fundamental concepts associated with differential equations. First, however, you are asked to recall some terminology and notation from your previous exposure to calculus.

Some of this terminology and notation was discussed in Unit 1.

The *derivative* of a variable y with respect to another variable x is denoted in Leibniz notation by dy/dx. In this expression we refer to y as the *dependent variable* and x as the *independent variable*.

Other notations are also used for derivatives. If the relation between variables x and y is expressed in terms of a function f, so that $y = f(x)$, then the derivative may be written in function notation as $f'(x)$. A further notation, attributed to Newton, is restricted to cases in which the independent variable is time, denoted by t. The derivative of $y = f(t)$ can be written in this case as $\dot{y}$, where the dot over the y stands for the d/dt of Leibniz notation.

Thus we may express this derivative in any of the equivalent forms

$$\frac{dy}{dt} = \dot{y} = f'(t).$$

Further derivatives are obtained by differentiating this first derivative. The second derivative of $y = f(t)$ could be represented by any of the forms

$$\frac{d^2y}{dt^2} = \ddot{y} = f''(t).$$

These possible notations have different strengths and weaknesses, and which is most appropriate in any situation depends on the purpose at hand. You will see all of these notations used at various times during the module.

It is common practice in science and applied mathematics to reduce the proliferation of symbols as far as possible. One aspect of this practice is that we often avoid allocating separate symbols to variables and to associated functions. So in place of the equation $y = f(t)$ (where y and t denote variables, and f denotes the function that relates them), we write $y = y(t)$, which is read as 'y is a function of t'. (You saw an example of this in the preceding subsection, where we used P and $P(t)$.)

Strictly speaking, this is an abuse of notation, since there is ambiguity as to exactly what the symbol y represents: it is a variable on the left-hand side of $y = y(t)$, but a function on the right-hand side. However, this is a very convenient abuse!

The following definitions explain just what are meant by a *differential equation*, by the *order* of such an equation, and by a *solution* of it.

Definitions

- A **differential equation** for $y = y(x)$ is an equation that relates the independent variable x, the dependent variable y, and one or more derivatives of y with respect to x.
- The **order** of a differential equation is the order of the highest derivative that appears in the equation. Thus a **first-order differential equation** for $y = y(x)$ features only the first derivative, dy/dx.
- A **solution** of a differential equation is a function $y = y(x)$ that satisfies the equation.

These definitions have been framed in terms of an independent variable x and a dependent variable y. You should be able to translate them to apply to any other independent and dependent variables. Thus equation (9) is a differential equation in which t is the independent variable and P is the dependent variable. It is a first-order equation, since dP/dt appears in it but higher derivatives such as d^2P/dt^2 do not. By contrast, the differential equation

$$3\,\frac{d^2y}{dx^2} + 2\,\frac{dy}{dx} + y^2 \sin x = x^2$$

is of second order, since the second derivative d^2y/dx^2 appears in it but higher derivatives do not. Second-order differential equations will be discussed in Unit 3.

The topic of this unit is first-order differential equations. Moreover, it concentrates on first-order equations that can be expressed (possibly after some algebraic manipulation) in the form

$$\frac{dy}{dx} = f(x, y). \tag{10}$$

Equation (9) is of this form, with $f(t,P) = kP\left(1 - \dfrac{P}{M}\right)$.

The right-hand side here stands for an expression involving both, either or neither of the variables x and y, but no other variables and no derivatives.

According to the definition given above, a function has to *satisfy* a differential equation to be regarded as a solution of it. The differential equation is satisfied by the function provided that when the function is substituted into the equation, the left- and right-hand sides of the equation give identical expressions.

This substitution includes the requirement that the function should be *differentiable* (i.e. that it should have a derivative) at all points where it is claimed to be a solution.

In the next exercise you are asked to verify that several functions are solutions of corresponding first-order differential equations. Later in the unit, you will see how all of these differential equations may be solved; but even when a solution has been deduced, it is worth checking by substitution that the supposed solution is indeed valid.

Exercise 2

Verify that each of the following functions is a solution of the corresponding differential equation.

(a) $y = 2e^x - (x^2 + 2x + 2)$; $\quad \dfrac{dy}{dx} = y + x^2$.

(b) $y = \frac{1}{2}x^2 + \frac{3}{2}$; $\quad \dfrac{dy}{dx} = x$.

(c) $u = 2e^{x^2/2}$; $\quad u' = xu$.

(d) $y = \sqrt{\dfrac{27 - x^2}{3}}$ $(-3\sqrt{3} < x < 3\sqrt{3})$; $\quad \dfrac{dy}{dx} = -\dfrac{x}{3y}$ $(y \neq 0)$.

(e) $y = t + e^{-t}$; $\quad \dot{y} = -y + t + 1$.

(f) $y = t + Ce^{-t}$; $\quad \dot{y} = -y + t + 1$. (Here C is an arbitrary constant.)

The restriction $y \neq 0$ placed on the differential equation in part (d) is necessary to ensure that $-x/3y$ is well defined.

In the last two parts of Exercise 2 you were asked to verify that

$$y = t + e^{-t} \quad \text{and} \quad y = t + Ce^{-t}$$

are solutions of the differential equation $\dot{y} = -y + t + 1$, where in the second case C is an *arbitrary constant*. Whatever number is chosen for C, the corresponding expression for $y(t)$ is always a solution of the differential equation. The particular function $y = t + e^{-t}$ is just one example of such a solution, obtained by choosing $C = 1$.

This demonstrates that solutions of a differential equation can exist in profusion; as a result, we need terms to distinguish between the totality of all the solutions for a given equation and an individual solution that is just one of the possibilities.

Definitions

- The **general solution** of a differential equation is the collection of all possible solutions of that equation.
- A **particular solution** of a differential equation is a single solution of the equation, and consists of a solution containing no arbitrary constants.

In many cases it is possible to describe the general solution of a first-order differential equation by a single formula involving one arbitrary constant. For example, $y = t + Ce^{-t}$ is the general solution of the equation $\dot{y} = -y + t + 1$; this means that not only is $y = t + Ce^{-t}$ a solution for all values of C, but also *every* particular solution of the equation may be obtained by giving C a suitable value.

Sometimes the values allowed for the arbitrary constant are restricted in some way. In the above example, if y is real-valued, we should take C to be a real (rather than a complex) number. In other cases, the general solution makes sense only if the arbitrary constant C is restricted to some range, and you will meet examples of this later in this unit. Nevertheless, any arbitrary constant is 'arbitrary' in the sense that it does not have a definite value.

Exercise 3

(a) Verify that, for any value of the constant C, the function $y = C - \frac{1}{3}e^{-3x}$ is a solution of the differential equation

$$\frac{dy}{dx} = e^{-3x}.$$

(b) Verify that, for any value of the constant C, the function $u = Ce^t - t - 1$ is a solution of the differential equation

$$\dot{u} = t + u.$$

(c) Verify that, for any value of the constant C, the function

$$P = \frac{CMe^{kt}}{1 + Ce^{kt}}$$

is a solution of equation (9).

As you have seen, there are many solutions of a differential equation. However, a particular solution of the equation, representing a definite relationship between the variables involved, is often what is needed. This is obtained by using a further piece of information in addition to the differential equation. Often the extra information takes the form of a pair of values for the independent and dependent variables.

For example, in the case of a population model, it would be natural to specify the starting population, P_0 say, and to start measuring time from $t = 0$. We could then write

$$P = P_0 \text{ when } t = 0, \quad \text{or equivalently,} \quad P(0) = P_0.$$

A requirement of this type is called an *initial condition*.

Definitions

- An **initial condition** associated with the differential equation

  $$\frac{dy}{dt} = f(t, y)$$

  specifies that the dependent variable y takes some value y_0 when the independent variable t takes some value t_0. This is written either as

  $$y = y_0 \text{ when } t = t_0$$

  or as

  $$y(t_0) = y_0.$$

  The numbers t_0 and y_0 are referred to as **initial values**.
- The combination of a first-order differential equation and an initial condition is called an **initial-value problem**.

The word 'initial' in these definitions arises from those (frequent) cases in which:

- the independent variable represents time t, and the differential equation describes how a system evolves in time
- the initial condition $y(t_0) = y_0$ describes the way the system is started off at some initial time t_0
- we are interested in how the system behaves at times after the initial time t_0, so we want the solution $y(t)$ for $t > t_0$.

However, this situation is not essential. The term 'initial condition' is used even when we are interested in the solution $y(t)$ for $t < t_0$ or when the independent variable does not represent time.

We usually expect that an initial-value problem should have a *unique* solution, since the outcome is then completely determined by the differential equation that governs the system, and by the configuration that the system had at the start. In fact, all the initial-value problems you will meet in this module have unique solutions, so if you can find a solution to an initial-value problem, then this is *the* solution.

Example 1

Using the result given in Exercise 3(b), find a solution to the initial-value problem

$$\frac{dy}{dx} = x + y, \quad y(0) = 1.$$

Solution

From Exercise 3(b), on replacing the variables t, u by x, y, the differential equation in this question has solutions of the form

$$y = Ce^x - x - 1.$$

The initial condition tells us that $y = 1$ when $x = 0$, and on feeding these values into the above solution we find that

$$1 = Ce^0 - 0 - 1 = C - 1.$$

Hence $C = 2$, and the particular solution of the differential equation that solves the initial-value problem is

$$y = 2e^x - x - 1.$$

Exercise 4

The size P of a population (measured in hundreds of thousands) is modelled by the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \quad P(0) = 1,$$

where $k = 0.15$, $M = 10$, and t is time measured in years.

(a) Use the result given in Exercise 3(c) to find a solution to this initial-value problem.

(b) Use your answer to predict the long-term behaviour of the population size (as $t \to \infty$).

Finally in this subsection, note that we sometimes need to keep an eye on the *domain* of the function solving the differential equation. 'Gaps' in the domain usually show up as some form of restriction on the nature of a solution curve. For example, consider the differential equation

$$\frac{dy}{dx} = \frac{1}{x}. \tag{11}$$

It turns out that there are two distinct families of solutions, illustrated in Figure 9. One family of solutions (given by $y = \ln x + C$) applies for $x > 0$, and another family (given by $y = \ln(-x) + C$) applies for $x < 0$. The right-hand side of the differential equation is not defined for $x = 0$, so there is no solution there, and the two families do not cross the y-axis.

Since $|x| = -x$ if $x < 0$, you can see that this agrees with what we know from Unit 1, namely that
$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

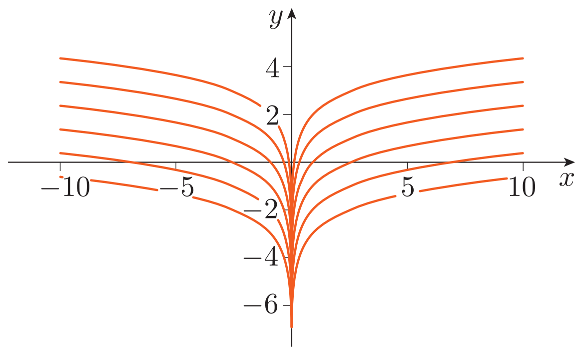


Figure 9 The two families of solutions, for $x > 0$ and $x < 0$, arising from equation (11)

2 Direct integration and separation of variables

This section and the next look at methods for finding *analytic* solutions of first-order differential equations – that is, solutions expressed in terms of exact formulas.

Until now we have considered first-order differential equations of the form

$$\frac{dy}{dx} = f(x, y). \qquad \text{(Eq. 10)}$$

We now specialise to two important cases where analytic solutions can be found. Each case involves a special technique. The method of *direct integration* applies to first-order differential equations of the form

$$\frac{dy}{dx} = g(x), \qquad (12)$$

where the right-hand side is a function of x alone. The method of *separation of variables* applies to first-order differential equations of the form

$$\frac{dy}{dx} = g(x)\,h(y), \qquad (13)$$

where the right-hand side is the *product* of a function of x and a function of y. Of course, equation (12) is a special case of equation (13) with $h(y) = 1$. Nevertheless, it is convenient to discuss equation (12) first because it is solved in a very straightforward way.

2.1 Direct integration

An example of a differential equation that can be solved by direct integration is

$$\frac{dy}{dx} = x^2. \qquad (14)$$

In order to solve this equation, we need to find functions $y(x)$ whose derivatives are x^2; one such function is $y = \frac{1}{3}x^3$. There are other functions with this same derivative, for example, $y = \frac{1}{3}x^3 + 1$ and $y = \frac{1}{3}x^3 - 2$. In fact, any function of the form

$$y = \tfrac{1}{3}x^3 + C, \tag{15}$$

The values $C = 0$, $C = 1$ and $C = -2$ give the three particular solutions mentioned above.

where C is an arbitrary constant, satisfies differential equation (14), and equation (15) is the general solution of this differential equation.

The expression $\frac{1}{3}x^3 + C$ is also *the indefinite integral* of x^2: that is,

$$\int x^2\,dx = \tfrac{1}{3}x^3 + C.$$

This is hardly surprising, since integration 'undoes' or reverses the effect of differentiation.

In this case, therefore, the indefinite integral of x^2 is the general solution of differential equation (14), and a similar connection applies more generally, as you will now see.

If we have a differential equation of the form

$$\frac{dy}{dx} = f(x), \tag{16}$$

The function $f(x)$ is assumed to be continuous (i.e. its graph has no breaks).

where the right-hand side, $f(x)$, is a known function of x alone, then we simply integrate both sides with respect to x:

$$\int \frac{dy}{dx}\,dx = \int f(x)\,dx.$$

The left-hand side gives y, so we get

$$y = \int f(x)\,dx = F(x) + C, \tag{17}$$

where $F(x)$ is an integral of $f(x)$, and C is an arbitrary constant.

This means that the *general solution* of equation (16) can be written down directly as an indefinite integral; and if the integration can be performed, then the equation is solved.

Procedure 1 Direct integration

The *general solution* of the differential equation

$$\frac{dy}{dx} = f(x) \qquad \text{(Eq. 16)}$$

is

$$y = \int f(x)\,dx = F(x) + C, \qquad \text{(Eq. 17)}$$

where $F(x)$ is an integral of $f(x)$, and C is an arbitrary constant.

Once the general solution has been found, it is possible to single out a particular solution by specifying a value for the constant C. This value can be found by applying an initial condition.

Example 2

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = e^{-3x}.$$

(b) Find the particular solution of this differential equation that satisfies the initial condition $y(0) = \frac{5}{3}$.

Solution

(a) On applying direct integration, we obtain the general solution

$$y = \int e^{-3x}\,dx = -\tfrac{1}{3}e^{-3x} + C,$$

where C is an arbitrary constant.

(b) The initial condition $y(0) = \frac{5}{3}$ tells us that $y = \frac{5}{3}$ when $x = 0$, so we must have

$$\tfrac{5}{3} = -\tfrac{1}{3}e^{0} + C,$$

which gives $C = 2$. The required particular solution is therefore

$$y = -\tfrac{1}{3}e^{-3x} + 2.$$

Procedure 1 uses x for the independent variable and y for the dependent variable. As usual, you should be prepared to translate this into situations where other symbols are used for the variables. But remember that the method of direct integration applies only to first-order differential equations for which the derivative is equal to a function of the *independent* variable alone. Thus direct integration can be applied, for example, to the differential equation

$$\frac{dx}{dt} = \cos t,$$

which has t as the independent variable and x as the dependent variable. In this case, the general solution is

$$x = \int \cos t\,dt = \sin t + C,$$

where C is an arbitrary constant. On the other hand, the differential equation

$$\frac{dx}{dt} = x^2$$

cannot be solved by direct integration because the right-hand side here is a function of the *dependent* variable, x.

Exercise 5

Solve each of the following initial-value problems.

(a) $\dfrac{dy}{dx} = 6x, \quad y(1) = 5.$

(b) $\dfrac{dv}{du} = e^{4u}, \quad v(0) = 2.$

(c) $\dot{y} = 5\sin 2t, \quad y(0) = 0.$

Remember that $\dot{y}$ stands for dy/dt, where t denotes time.

The method of direct integration succeeds in solving a differential equation of the specified type whenever it is possible to carry out the integration that arises – a task that may require you to apply any of the standard techniques of integration, such as the use of tables, integration by parts or integration by substitution (see Unit 1). For more difficult integrals, a computer algebra package may be used.

Exercise 6

Find the general solution of each of the following differential equations.

(a) $\dfrac{dy}{dx} = xe^{-2x}$ (*Hint*: For the integral, try integration by parts.)

(b) $\dot{p} = \dfrac{t}{1+t^2}$ (*Hint*: For the integral, try the substitution $u = 1 + t^2$.)

The solution to Exercise 6(b) can be generalised to any differential equation of the form

$$\frac{dy}{dx} = k\,\frac{f'(x)}{f(x)} \quad (\text{for } f(x) \neq 0),$$

where k is a constant. We get the general solution

$$y = k\ln|f(x)| + C,$$

where C is an arbitrary constant.

This is a simple extension of the result from Unit 1 that

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C,$$

for $f(x) \neq 0$.

2.2 Separation of variables

Direct integration applies, in an immediate sense, only to the very simplest type of differential equation, as described by equation (16). However, all other analytic methods of solution for first-order equations also eventually boil down to performing integrations. In this subsection, we consider how to solve first-order differential equations of the form

$$\frac{dy}{dx} = g(x)\,h(y), \tag{18}$$

where the right-hand side is a product of a function of x and a function of y. Such first-order differential equations are said to be **separable**.

An easy example

Let us start with the simple example that we solved in the Introduction, namely

$$\frac{dy}{dx} = -Ay, \tag{19}$$

where $y(x) > 0$. This equation is of the required form, with $g(x) = 1$ and $h(y) = -Ay$. In the Introduction we obtained the solution by guesswork, but we will now use a systematic method to find the solution.

Dividing both sides of equation (19) by y, we obtain

Because $y > 0$, division by y causes no problems.

$$\frac{1}{y}\frac{dy}{dx} = -A.$$

Integrating both sides then gives

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = -A\int dx.$$

See equation (81) in Subsection 6.3 of Unit 1.

Applying the rule for integration by substitution to the left-hand side, we obtain

$$\int \frac{1}{y}\,dy = -A\int dx.$$

Integrating both sides (and remembering that $y > 0$) then gives

$$\ln y + C_1 = C_2 - Ax,$$

where C_1 and C_2 are constants of integration. We can rearrange this equation, and combine the constants, to obtain

$$\ln y = C - Ax$$

for some constant $C = C_2 - C_1$. Taking exponentials on both sides, we get

$$y = \exp(C - Ax) = \exp(C)\exp(-Ax) = B\exp(-Ax),$$

where $B = \exp(C)$ is another arbitrary constant. This is the same as the solution obtained (by guesswork) in the Introduction.

In this case, the arbitrary constant B is restricted to positive values because $\exp(C) > 0$ for all C. (This restriction is related to our initial requirement that y is positive.) Note also that B is not simply added to the solution in this case.

A more typical example

Another example of a separable differential equation, as specified in equation (18), is

$$\frac{dy}{dx} = x(1 + y^2). \tag{20}$$

Here, the right-hand side is a product of the functions $g(x) = x$ and $h(y) = 1 + y^2$. You are unlikely to guess the solution in this case. Here we develop the systematic *separation of variables method* for its solution,

so-called because we rewrite the equation in a form where only y appears on one side, and only x on the other.

As a first step, we divide both sides of this equation by $1+y^2$, to obtain

Note that $1+y^2$ is never zero, so it is safe to divide by it.

$$\frac{1}{1+y^2}\,\frac{dy}{dx}=x,$$

and then integrate both sides with respect to x, to get

$$\int \frac{1}{1+y^2}\,\frac{dy}{dx}\,dx=\int x\,dx. \tag{21}$$

The rule for integration by substitution tells us that the left-hand side of this equation can be rewritten as

See Section 6 of Unit 1.

$$\int \frac{1}{1+y^2}\,\frac{dy}{dx}\,dx=\int \frac{1}{1+y^2}\,dy,$$

so equation (21) becomes

$$\int \frac{1}{1+y^2}\,dy=\int x\,dx.$$

At this point, we have achieved the desired separation: the left-hand side depends only on y, and the right-hand side depends only on x. On performing the two integrations (using the table of standard integrals in the Handbook) we obtain

$$\arctan y=\tfrac{1}{2}x^2+C, \tag{22}$$

where C is an arbitrary constant. Making y the subject of the equation, we obtain the solution

Note that one arbitrary constant is sufficient here.

$$y(x)=\tan\left(\tfrac{1}{2}x^2+C\right). \tag{23}$$

General procedure

The approach just demonstrated applies more widely. In principle, it works for any differential equation of the form

$$\frac{dy}{dx}=g(x)\,h(y). \qquad \text{(Eq. 18)}$$

On dividing both sides of this equation by $h(y)$ (for all values of y other than those where $h(y)=0$), we obtain

$$\frac{1}{h(y)}\,\frac{dy}{dx}=g(x).$$

Integration with respect to x on both sides then gives

$$\int \frac{1}{h(y)}\,\frac{dy}{dx}\,dx=\int g(x)\,dx,$$

and, on applying the rule for integration by substitution to the left-hand side, this becomes

$$\int \frac{1}{h(y)}\,dy=\int g(x)\,dx. \tag{24}$$

This is the form that you need to remember!

If the two integrals can be evaluated at this stage, we obtain an equation that relates x and y, and features an arbitrary constant. This equation is the general solution of the differential equation (for values of y other than those where $h(y) = 0$). Usually, however, y is not the subject of this equation. Such a solution is said to be an **implicit general solution** of the differential equation: an example is provided by equation (22).

Usually, the final aim is to make y the subject of the equation, if possible – that is, to manipulate the equation into the form

$$y = \text{function of } x.$$

This is called the **explicit general solution** of the differential equation: an example is provided by equation (23).

In either case (implicit or explicit), a particular solution can be obtained from the general solution by applying an initial condition.

The method just described for solving differential equations of the form (18) is called the method of *separation of variables* since, in equation (24), we have separated the variables to either side of the equation, with only the dependent variable appearing on the left, and only the independent variable on the right.

So far, we have assumed that $h(y) \neq 0$. However, the condition $h(y) = 0$ corresponds to $dy/dx = g(x)\,h(y) = 0$ for all x, and so gives extra solutions to the differential equation with $y = \text{constant}$. These exceptional cases should be included in the general solution. The method is summarised below.

Procedure 2 Separation of variables

This method applies to **separable differential equations**, which are of the form

$$\frac{dy}{dx} = g(x)\,h(y). \qquad \text{(Eq. 18)}$$

1. Assume that $h(y) \neq 0$, and divide both sides of the differential equation by $h(y)$. Integrate both sides with respect to x. The rule for integration by substitution then gives
$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx. \qquad \text{(Eq. 24)}$$
2. If possible, perform the integrations, including an arbitrary constant.
3. If possible, rearrange the formula found in Step 2 to give a family of solutions $y(x)$, including an arbitrary constant.
4. Find any constant values of y for which $h(y) = 0$.
5. The *general solution* is given by the family of solutions in Step 3, supplemented by any constant solutions found in Step 4.

It is a good idea to check, by substitution into the original differential equation, that the function obtained is indeed a solution.

The method of separation of variables is useful, but there may be some difficulties. First, it may not be possible to perform the necessary integrations. Second, it may not be possible to perform the necessary manipulations to obtain an explicit solution.

It is sometimes necessary to be careful about the *domain* of the solution obtained, as the following example illustrates.

Example 3

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{3y} \quad (y > 0).$$

(b) Find the particular solution that satisfies the initial condition $y(0) = 3$.

Solution

(a) The equation is of the form

$$\frac{dy}{dx} = g(x)\,h(y),$$

where the obvious choices for g and h are

$$g(x) = -x \quad \text{and} \quad h(y) = 1/(3y).$$

Notice that $h(y)$ is never equal to zero.

We now apply Procedure 2. On dividing through by $h(y) = 1/(3y)$ (i.e. multiplying through by $3y$) and integrating with respect to x, the differential equation becomes

$$\int 3y\,dy = \int -x\,dx.$$

With practice, you will be able to move directly to this stage, as shown in Procedure 2.

Evaluating the integrals gives

$$\tfrac{3}{2}y^2 = -\tfrac{1}{2}x^2 + C,$$

where C is an arbitrary constant. This is an implicit form of the general solution.

On solving for y and noting the condition $y > 0$ given in the question (which determines the sign of the square root), we obtain

$$y = \sqrt{\tfrac{1}{3}(2C - x^2)}.$$

This can be simplified slightly by writing B in place of $2C$. So

$$y = \sqrt{\tfrac{1}{3}(B - x^2)},$$

where B is an arbitrary constant. In fact, the value of B is somewhat restricted. This is because the condition $y > 0$ implies that y is real, and this requires that $B - x^2 > 0$. Hence $B > x^2 \geq 0$, which tells us that B must be restricted to positive values. At the same time, the condition $B - x^2 > 0$ restricts the possible values of x to lie between $-\sqrt{B}$ and $\sqrt{B}$.

The explicit general solution in this case is therefore

$$y = \sqrt{\tfrac{1}{3}(B - x^2)} \quad (-\sqrt{B} < x < \sqrt{B}),$$

where B is any positive constant.

(b) The initial condition is $y(0) = 3$, so we substitute $x = 0$ and $y = 3$ into the general solution above. This gives $3 = \sqrt{\tfrac{1}{3}B}$, so $B = 27$, and the required particular solution is

You verified in Exercise 2(d) that this function is a solution of the differential equation.

$$y(x) = \sqrt{\tfrac{1}{3}(27 - x^2)} \quad (-3\sqrt{3} < x < 3\sqrt{3}).$$

Here are some exercises on applying the method of separation of variables. Remember the following points:

- The method initially assumes that $h(y) \neq 0$ and then gives a family of solutions containing an arbitrary constant.
- The condition $h(y) = 0$ may give extra solutions $y =$ constant. These should be included in the general solution; they may or may not have the same form as the general family of solutions obtained for $h(y) \neq 0$.

When doing these exercises, try to obtain the *general* solution, including all cases where $h(y) = 0$.

Exercise 7

Find the general solution of each of the following differential equations.

(a) $\dfrac{dy}{dx} = \dfrac{y-1}{x} \quad (x > 0)$ (b) $\dfrac{dy}{dx} = \dfrac{2y}{1+x^2}$

Exercise 8

Solve the initial-value problem

$$\frac{dv}{du} = e^{u+v}, \quad v(0) = 0.$$

Exercise 9

Find the general solution of each of the following differential equations.

(a) $u' = xu$ (b) $\dot{x} = 1 + x^2$

Exercise 10

The differential equation here is the logistic equation (equation (9)), which is used to model population sizes.

(a) Solve the initial-value problem

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \quad \text{with } P(0) = 2M,$$

where $P(t) > 0$, and k and M are positive constants.

You may use the fact that

$$\int \frac{1}{x(1-ax)}\,dx = -\ln\left|\frac{1}{x} - a\right| + C \quad \text{for } x \neq 1/a,$$

where C is an arbitrary constant of integration.

(b) Describe what happens to the solution $P(t)$ as t becomes large and positive.

3 Solving linear differential equations

This section presents one final method for finding analytic solutions of first-order differential equations. The method, called the *integrating factor method*, applies only to a particular form of equation known as a *linear* differential equation. The definition and some properties of this type of equation are introduced in Subsection 3.1. Subsection 3.2 shows how to solve a particularly simple type of linear differential equation before we move on to the general method of solution in Subsection 3.3.

3.1 Linear differential equations

This subsection introduces the concept of *linearity* as applied to differential equations. Here the concept is introduced in the context of first-order differential equations, but you should be aware that the idea generalises to higher-order differential equations and is important from a theoretical point of view.

Linear second-order differential equations are considered in Unit 3.

Definitions

- A first-order differential equation for $y = y(x)$ is **linear** if it can be expressed in the form

$$\frac{dy}{dx} + g(x)\,y = h(x), \tag{25}$$

  where $g(x)$ and $h(x)$ are given functions.
- A linear first-order differential equation is said to be **homogeneous** if $h(x) = 0$ for all x, and **inhomogeneous** or **non-homogeneous** otherwise.

This differential equation is of the general form

$$\frac{dy}{dx} = f(x, y)$$

used elsewhere in this unit, with

$$f(x, y) = -g(x)\,y + h(x).$$

Note that linearity refers to the dependent variable y. Terms like y^2 or $y\,(dy/dx)$ are excluded, but any function of the independent variable x is allowed. So, for example, the differential equation

$$\frac{dy}{dx} - x^2 y = x^3$$

is linear, with $g(x) = -x^2$ and $h(x) = x^3$, whereas the equation

$$\frac{dy}{dx} = xy^2$$

is not, due to the presence of the non-linear term y^2.

It can be shown that any initial-value problem based on equation (25) has a unique solution provided that $g(x)$ and $h(x)$ are continuous functions.

Exercise 11

Decide whether or not each of the following first-order differential equations is linear.

(a) $\dfrac{dy}{dx} + x^3y = x^5$ (b) $\dfrac{dy}{dx} = x\sin x$ (c) $\dfrac{dz}{dt} = -3z^{1/2}$

(d) $\dot{y} + y^2 = t$ (e) $x\dfrac{dy}{dx} + y = y^2$ (f) $(1+x^2)\dfrac{dy}{dx} + 2xy = 3x^2$

3.2 Linear constant-coefficient equations

The simplest example of a linear equation occurs when the function $g(x)$ does not depend on x, that is, where the coefficient of y is constant. In the case of a homogeneous equation, this is just the simple example that we dealt with in the Introduction: if $g(x) = A$, then the homogeneous equation is

In equation (25), $g(x)$ is the coefficient of y.

$$\frac{dy}{dx} + Ay = 0,$$

which has solution $y = y_0 \exp(-Ax)$, as you saw in Section 2.

Now consider the case of the *inhomogeneous* linear differential equation with constant coefficients, which is of the form

Equations of this type are met in the theory of electrical circuits.

$$\frac{dy}{dx} + Ay = h(x). \tag{26}$$

We will now use a trick to solve this equation, which at first sight is not obvious. We begin by multiplying both sides by e^{Ax}:

$$e^{Ax}\frac{dy}{dx} + Ae^{Ax}y = e^{Ax}\,h(x). \tag{27}$$

Now, using the product rule for differentiation, we notice that

$$\frac{d}{dx}\big(e^{Ax}y\big) = e^{Ax}\frac{dy}{dx} + Ae^{Ax}y.$$

This means that the left-hand side of equation (27) can be written as $\dfrac{d}{dx}\big(e^{Ax}y\big)$, so we have

$$\frac{d}{dx}\big(e^{Ax}y\big) = e^{Ax}\,h(x). \tag{28}$$

This differential equation can now be solved by integrating both sides with respect to x and rearranging the result.

It is easier to see how this works by considering particular examples, but for the record, the general method proceeds as follows. Integration gives

$$\int \frac{d}{dx}\left(e^{Ax}y\right)dx = \int e^{Ax}\,h(x)\,dx,$$

which becomes

$$e^{Ax}y = \int e^{Ax}\,h(x)\,dx,$$

We can omit the arbitrary constant of integration here because the remaining indefinite integral will still generate one.

so

$$y(x) = e^{-Ax}\left(\int e^{Ax}\,h(x)\,dx\right). \tag{29}$$

There is no difficulty in dividing through by e^{Ax} because this is never equal to zero.

This shows that if we can perform the integral on the right-hand side, then we can solve our original differential equation (26). The integral will generate an arbitrary constant, and our solution will be the general solution of the differential equation.

Example 4

Solve the linear differential equation

$$\frac{dy}{dx} + 2y = x.$$

Solution

The differential equation has the same form as equation (26), with $A = 2$ and $h(x) = x$, so we must multiply by the factor e^{2x}. Multiplying both sides of the differential equation by e^{2x}, we get

$$e^{2x}\frac{dy}{dx} + e^{2x}2y = e^{2x}x.$$

Following equation (28), the differential equation can then be expressed in the form

$$\frac{d}{dx}\left(e^{2x}y\right) = e^{2x}x.$$

Integrating both sides gives

$$e^{2x}y = \int xe^{2x}\,dx.$$

The integral on the right-hand side can be evaluated by parts, using the formula

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int g(x)\,f'(x)\,dx.$$

Choosing $f(x) = x$ and $g'(x) = e^{2x}$, we have $f'(x) = 1$ and $g(x) = \frac{1}{2}e^{2x}$, so

$$\begin{aligned} e^{2x}y &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}\,dx \\ &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\left(\tfrac{1}{2}e^{2x} + C\right) \\ &= \tfrac{1}{2}e^{2x}\left(x - \tfrac{1}{2}\right) - \tfrac{1}{2}C. \end{aligned}$$

Thus

$$y(x) = \tfrac{1}{2}\left(x - \tfrac{1}{2} - Ce^{-2x}\right),$$

where C is an arbitrary constant.

We can check this answer by differentiation:

$$\frac{dy}{dx} + 2y = \tfrac{1}{2}\left(1 + 2Ce^{-2x}\right) + \left(x - \tfrac{1}{2} - Ce^{-2x}\right) = x,$$

as required.

Exercise 12

Solve the differential equation

$$\frac{dy}{dx} + y = e^{2x}$$

with the initial condition $y(0) = 0$. Check your solution by differentiation.

You have seen how to solve linear differential equations with constant coefficients. This method is easily generalised to the case where the coefficient is not constant. The general form of the technique is called the *integrating factor* method.

3.3 The integrating factor method

The secret to solving the linear constant-coefficient differential equation

$$\frac{dy}{dx} + Ay = h(x)$$

is to multiply both sides by e^{Ax} and to notice that

$$\frac{d}{dx}\left(e^{Ax}y\right) = e^{Ax}\frac{dy}{dx} + e^{Ax}Ay,$$

which is e^{Ax} times the left-hand side of the differential equation.

We will now try something very similar for the general linear differential equation

$$\frac{dy}{dx} + g(x)\,y = h(x). \qquad \text{(Eq. 25)}$$

In this case, we multiply both sides by $e^{G(x)}$, where the function $G(x)$ remains to be decided. This gives

$$e^{G(x)}\frac{dy}{dx} + e^{G(x)}\,g(x)\,y = e^{G(x)}\,h(x). \qquad (30)$$

We then notice that the chain rule of differentiation gives

$$\frac{d}{dx}\left(e^{G(x)}\,y\right) = e^{G(x)}\frac{dy}{dx} + e^{G(x)}\frac{dG}{dx}\,y.$$

This can be made equal to $e^{G(x)}$ times the left-hand side of the differential equation (25) *provided that* we take

$$\frac{dG}{dx} = g(x), \tag{31}$$

and this implies that

$$G(x) = \int g(x)\,dx.$$

We can omit the arbitary constant when evaluating this integral. This is because we need only one function $G(x)$ that satisfies equation (31). We do not need the general function to do so.

With this choice of $G(x)$, equation (30) can be written as

$$\frac{d}{dx}\left(e^{G(x)}\,y\right) = e^{G(x)}\,h(x),$$

and the solution of the differential equation is then obtained by integrating both sides and rearranging the result.

You will soon see how this works in explicit examples. For the moment, the key point is that we need to multiply both sides of the equation by the factor

$$p(x) = e^{G(x)} = \exp\left(\int g(x)\,dx\right). \tag{32}$$

This factor is called the **integrating factor** of the differential equation because it allows the left-hand side of the equation to be integrated exactly. This leads to the following procedure.

Procedure 3 The integrating factor method

This method applies to any first-order *linear* differential equation, that is, any equation of the form

$$\frac{dy}{dx} + g(x)\,y = h(x). \tag{Eq. 25}$$

1. Determine the integrating factor

   $$p(x) = \exp\left(\int g(x)\,dx\right). \tag{Eq. 32}$$

2. Multiply equation (25) by $p(x)$ to recast the differential equation as

   $$p(x)\,\frac{dy}{dx} + p(x)\,g(x)\,y = p(x)\,h(x).$$

3. Rewrite the differential equation as

   $$\frac{d}{dx}(p(x)\,y) = p(x)\,h(x). \tag{33}$$

4. Integrate both sides of this equation, including an arbitrary constant, and rearrange the result to make $y(x)$ the subject of the equation. The result is the *general solution* of the differential equation.

The constant of integration is not needed here as it can be shown to cancel out in Step 3.

It is a good idea to check that the derivative on the left-hand side reproduces the left-hand side of the preceding equation.

Let us first apply this procedure to the constant coefficient equation (26), to check that it gives the correct solution (29).

Example 5

Use the integrating factor method in Procedure 3 to solve

$$\frac{dy}{dx} + Ay = h(x),$$

where A is a constant.

Solution

Comparing the equation in the question with equation (25), we see that $g(x) = A$. So the integrating factor, equation (32), is

$$p(x) = \exp\left(\int A\,dx\right) = \exp(Ax).$$

This is exactly the integrating factor that we used in Subsection 3.2.

Equation (33) then gives

$$\frac{d}{dx}\left(e^{Ax}y\right) = e^{Ax}\,h(x),$$

which is the same as equation (28). The remainder of the procedure follows the path that led to equation (29), so we get the solution

$$y(x) = e^{-Ax}\left(\int e^{Ax}\,h(x)\right).$$

As with the separation of variables method, it may not be possible to perform the necessary final integration. However, the remainder of this subsection gives examples and exercises for which the integrals can be done. It is important to note that the constant of integration must be included in the final integration, as this is what makes the solution a general one.

Example 6

Use the integrating factor method to find the general solution of the differential equation

$$\frac{dy}{dx} = x - \frac{2xy}{x^2+1}.$$

Solution

On rearranging the differential equation as

$$\frac{dy}{dx} + \frac{2xy}{x^2+1} = x,$$

we see that it has the form of a linear differential equation $dy/dx + g(x)\,y = h(x)$, with

$$g(x) = \frac{2x}{x^2+1} \quad \text{and} \quad h(x) = x.$$

The integrating factor (from equation (32)) is therefore

$$p(x) = \exp\left(\int \frac{2x}{x^2+1}\,dx\right).$$

Notice that $2x/(x^2+1)$ is of the form f'/f, so

$$\begin{aligned}\int \frac{2x}{x^2+1}\,dx &= \ln|x^2+1| \\ &= \ln(x^2+1) \quad \text{since } x^2+1>0.\end{aligned}$$

Remember that we do not need to include an arbitrary constant when finding an integrating factor.

The integrating factor is therefore

$$p(x) = \exp(\ln(x^2+1)) = x^2+1.$$

Multiplying both sides of the differential equation by this factor yields

$$(x^2+1)\,\frac{dy}{dx} + 2xy = x(x^2+1),$$

and the differential equation becomes

$$\frac{d}{dx}\big((x^2+1)y\big) = x(x^2+1).$$

Integrating both sides gives

$$\begin{aligned}(x^2+1)y &= \int x(x^2+1)\,dx \\ &= \int (x^3+x)\,dx \\ &= \tfrac{1}{4}x^4 + \tfrac{1}{2}x^2 + C,\end{aligned}$$

where C is an arbitrary constant. Finally, to obtain an explicit solution, we divide by x^2+1 to get

$$y(x) = \frac{x^4+2x^2+B}{4(x^2+1)},$$

where $B = 4C$ is an arbitrary constant.

The next example contains two differential equations that were solved in Exercise 7 using the method of separation of variables. Here, we solve them using the integrating factor method. You can compare the answers with those obtained earlier.

Example 7

Use the integrating factor method to find the general solution of each of the following differential equations.

(a) $\dfrac{dy}{dx} = \dfrac{y-1}{x} \quad (x>0)$

(b) $\dfrac{dy}{dx} = \dfrac{2y}{1+x^2}$

Solution

(a) On rearranging the differential equation as

$$\frac{dy}{dx} - \frac{1}{x}y = -\frac{1}{x}, \tag{34}$$

we see that it is of the linear form $dy/dx + g(x)\,y = h(x)$ with

$$g(x) = h(x) = -\frac{1}{x}.$$

The integrating factor (from equation (32)) is therefore

Recall that $a \ln x = \ln(x^a)$.

$$\begin{aligned} p(x) &= \exp\left(\int \left(-\frac{1}{x}\right) dx\right) \\ &= \exp(-\ln x) \quad (\text{since } x > 0) \\ &= \exp\left(\ln x^{-1}\right) \\ &= x^{-1} = \frac{1}{x}. \end{aligned}$$

Multiplying equation (34) by $p(x) = 1/x$ (and recalling that $x \neq 0$) gives

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}\,y = -\frac{1}{x^2}, \tag{35}$$

so the differential equation becomes

It is always worth a quick check at this stage: is the left-hand side of equation (36) equal to the left-hand side of equation (35)?

$$\frac{d}{dx}\left(\frac{1}{x}\,y\right) = -\frac{1}{x^2}. \tag{36}$$

Integration then gives

$$\frac{y}{x} = \int \left(-\frac{1}{x^2}\right) dx = \frac{1}{x} + C,$$

where C is an arbitrary constant. The general solution is therefore

$$y = 1 + Cx,$$

where C is an arbitrary constant.

(b) In order to put the given differential equation into the linear form $dy/dx + g(x)\,y = h(x)$, we need to bring the term in y to the left-hand side to obtain

$$\frac{dy}{dx} - \frac{2}{1+x^2}\,y = 0. \tag{37}$$

The equation is homogeneous.

Hence in this case we have $g(x) = -2/(1+x^2)$ and $h(x) = 0$.

To find the integrating factor, we must evaluate the integral

$$\int g(x)\,dx = \int \left(-\frac{2}{1+x^2}\right) dx = -2\arctan x,$$

so the integrating factor is

$$\exp\left(\int g(x)\,dx\right) = \exp(-2\arctan x) = e^{-2\arctan x}.$$

Multiplying through by this factor gives

$$e^{-2\arctan x}\frac{dy}{dx} - e^{-2\arctan x}\frac{2y}{1+x^2} = 0.$$

Thus the differential equation can be rewritten as

$$\frac{d}{dx}\left(e^{-2\arctan x}y\right) = 0.$$

You can check that the derivative on the left gives the left-hand side of the previous equation (using the fact that $d(\arctan x)/dx = 1/(1+x^2)$).

It follows, on integrating, that

$$e^{-2\arctan x}y = C, \quad \text{or equivalently,} \quad y(x) = Ce^{2\arctan x},$$

where C is an arbitrary constant. This is the general solution of the differential equation.

Exercise 13

Find the general solution of each of the following differential equations.

(a) $\dfrac{dy}{dx} - y = e^x \sin x$ (b) $\dfrac{dy}{dx} = y + x$

Exercise 14

Use the integrating factor method to solve each of the following initial-value problems.

(a) $u' = xu, \quad u(0) = 2.$

(b) $t\dot{y} + 2y = t^2, \quad y(1) = 1$ and $t > 0$.

You saw the differential equation in part (a) in Exercise 9(a), where you solved it using separation of variables.

Exercise 15

Solve each of the following initial-value problems.

(a) $\dot{y} + y = t + 1, \quad y(1) = 0.$

(b) $e^{3t}\dot{y} = 1 - e^{3t}y, \quad y(0) = 3.$

Exercise 16

Find the general solution of each of the following differential equations.

(a) $x\dfrac{dy}{dx} - 3y = x \quad (x > 0)$

(b) $\dfrac{dv}{dt} + 4v = 3\cos 2t$

(*Hint*: If a and b are non-zero constants, then

$$\int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt) + C,$$

where C is an arbitrary constant.)

You have now used a number of methods for solving first-order differential equations: direct integration, separation of variables and the integrating factor. Confronted with a fresh equation, the first issue is which method to try. The following exercise closes this discussion of analytic solutions by giving you practice at choosing an appropriate method.

Exercise 17

Which method(s) could you use to try to solve each of the following linear first-order differential equations? (You need not actually solve the equations.)

(a) $\dfrac{dy}{dx} + x^3 y = x^5$ (b) $\dfrac{dy}{dx} = x \sin x$ (c) $\dfrac{dv}{du} + 5v = 0$

(d) $(1 + x^2)\dfrac{dy}{dx} + 2xy = 1 + x^2$ (e) $\dfrac{dy}{dx} = y^2(1 + x^2)$

---

4 Direction fields

> The material in this section is non-assessable and will not be tested in continuous assessment or in the exam. However, you are advised to read this section and attempt the exercise, as this will provide valuable insights into the behaviour of solutions of differential equations.

Many of the differential equations that arise in physics and applied mathematics cannot be solved exactly. For this reason it is valuable to know about methods for obtaining approximate or qualitative information about solutions. It is also important to be aware of how computers may be used to give *numerical* solutions of differential equations, and this will be considered in Section 5.

In this section we show how qualitative information about the solutions of a first-order differential equation can be gleaned directly from the equation itself, without undertaking any form of integration process. The main concept here is the *direction field*, sketches of which usually give a good idea of how the graphs of solutions behave.

We start by considering what can be deduced about solutions of any differential equation of the form

$$\frac{dy}{dx} = f(x, y) \qquad \text{(Eq. 10)}$$

from direct observation of the equation.

In Subsection 1.1 we encountered the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \qquad \text{(Eq. 9)}$$

Here we have

$$f(t,P) = kP\left(1 - \frac{P}{M}\right).$$

where k and M are positive constants. In certain circumstances this is a useful mathematical model of population sizes, in which $P(t)$ denotes the size of the population at time t. The constant functions $P(t) = 0$ and $P(t) = M$ are particular solutions of the differential equation. This is because they both give $dP/dt = 0$ on the left-hand side and zero on the right-hand side, so the differential equation is satisfied in the form $0 = 0$. Within the population model, these solutions correspond to a complete absence of the population ($P = 0$), and an equilibrium population level ($P = M$) for which the birth rate exactly balances the death rate.

Spotting constant functions that are particular solutions of the differential equation is occasionally useful, but is of limited applicability. In general, more useful information can be deduced by noting that any particular solution $y(x)$ can be plotted as a graph of y against x in the xy-plane. At a given point (x_0, y_0), the gradient (or slope) of this graph is given by the value of dy/dx at (x_0, y_0), and according to the differential equation (10), this is given by $f(x_0, y_0)$. We therefore conclude that $f(x, y)$ represents the slope at (x, y) of a solution curve that passes through (x, y).

For example, if

$$\frac{dy}{dx} = f(x,y) = x + y, \qquad (38)$$

then the slope of a solution curve that passes through the point $(1, 2)$ is $f(1, 2) = 1 + 2 = 3$. This slope is positive, so the graph is increasing from left to right through the point $(1, 2)$. The slope of a (different) solution curve that passes through $(2, -7)$ is $f(2, -7) = 2 - 7 = -5$. This slope is negative, so the corresponding graph is decreasing from left to right through the point $(2, -7)$. At the point $(3, -3)$, the slope is $f(3, -3) = 3 - 3 = 0$, so the solution curve that passes through this point is horizontal.

When looking at $f(x, y)$ in this way, it is referred to as a *direction field*, since it describes a *direction* (or slope) for each point (x, y) where $f(x, y)$ is defined.

Definition

A **direction field** associates a unique direction to each point within a specified region of the xy-plane. The direction associated with the point (x, y) can be indicated by drawing a short line segment through the point, with the appropriate slope. This can be regarded as a segment of the tangent line at (x, y) of the solution curve that passes through (x, y).

In particular, the direction field for the differential equation

$$\frac{dy}{dx} = f(x, y)$$

associates the direction $f(x, y)$ with the point (x, y). This is the slope at (x, y) of the graph of a particular solution $y(x)$ that passes through the point (x, y).

Direction fields can be visualised by constructing the short line segments at a finite set of points in an appropriate region of the plane, where typically the points are chosen to form a rectangular grid. An example is shown in Figure 10, which corresponds to the differential equation (38). In this case the chosen region is the set of points (x, y) such that $-2 \leq x \leq 2$ and $0 \leq y \leq 2$, and the rectangular grid consists of the points at intervals of 0.2 in both the x- and y-directions within this region.

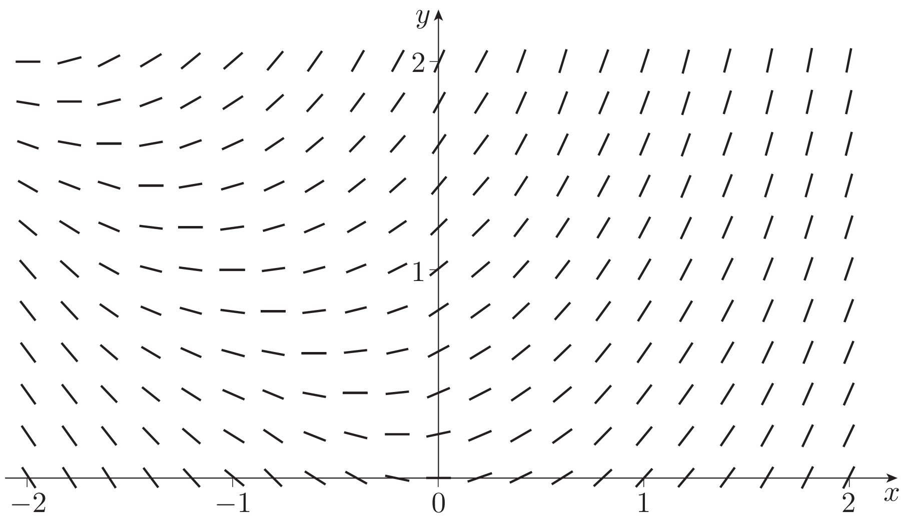


Figure 10 Part of the direction field for equation (38)

From this diagram, we can gain a good qualitative impression of how the graphs of particular solutions of equation (38) behave. The aim is to sketch curves on the diagram in such a way that the tangents to the curves are always parallel to the local slopes of the direction field. For example, starting from the point $(-1, 0.5)$ (that is, taking the initial condition to be $y(-1) = 0.5$), we expect the solution graph initially to fall as we move to the right. The magnitude of the negative slope decreases, however, and eventually reaches zero, after which the slope becomes positive and then increases. On this basis, we could sketch the graph of the corresponding particular solution and obtain something like the curve shown in Figure 11.

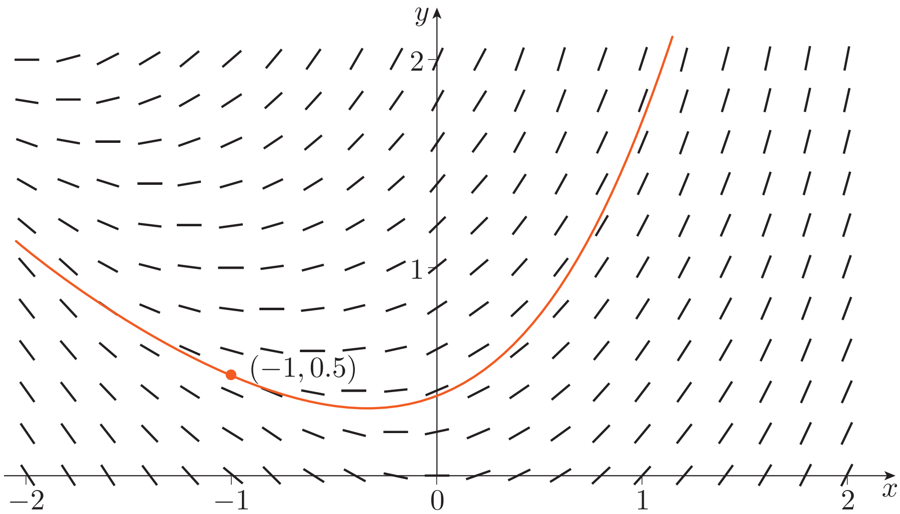


Figure 11 Part of the direction field for equation (38), and the particular solution satisfying $y(-1) = 0.5$ that passes through the point $(-1, 0.5)$

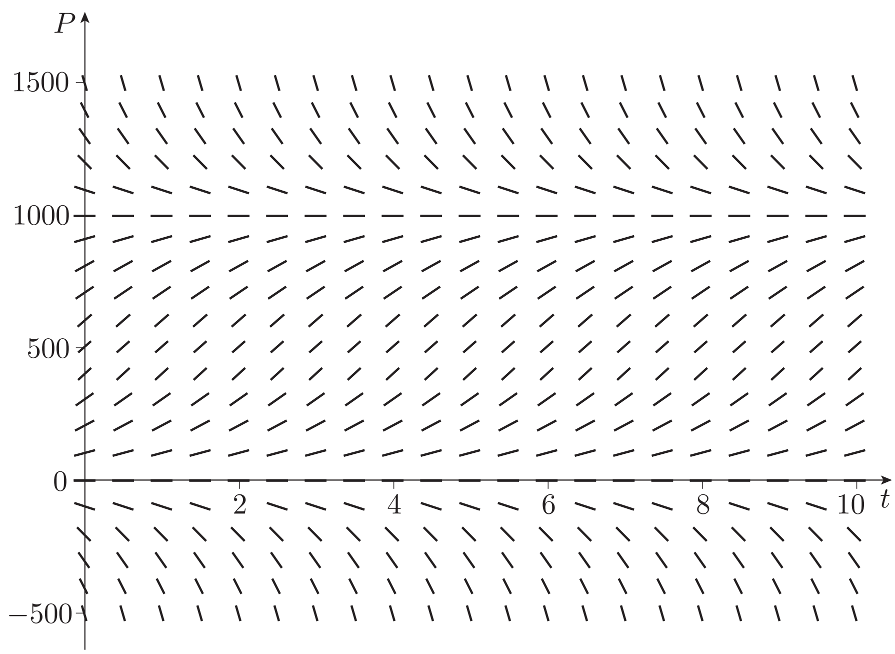
Exercise 18

(a) Part of the direction field for the logistic equation

$$\frac{dP}{dt} = P\left(1 - \frac{P}{1000}\right)$$

This is equation (9) with $k = 1$ and $M = 1000$.

is sketched in the figure below.



Using this diagram, sketch the solution curves that pass through the following points:

$$(0, 1500),\ (0, 1000),\ (0, 100),\ (0, 0),\ (0, -100).$$

From your results, describe the graphs of particular solutions of the differential equation.

(b) What does your answer to part (a) tell you about the predicted behaviour of a population whose size $P(t)$ at time t is modelled by this logistic equation?

When analysing problems involving differential equations it is sometimes qualitative information that is most important. For example, if we model the population size $P(t)$ of a species, it may be interesting to know whether the species dies out (does $P(t) \to 0$ as $t \to \infty$?) rather than having an accurate expression for P at a given value of t. Often the direction field approach gives direct access to insights that would be hard to extract from exact solutions. It can also be applied to differential equations for which no exact solutions can be found.

5 Numerical solutions and Euler's method

The material in this section is non-assessable and will not be tested in continuous assessment or in the exam. If you are running short of time, you can skip this section.

Numerical analysis

The method described here will give you a glimpse into a branch of mathematics called *numerical analysis*, which specialises in solving problems numerically on a computer. The problems extend beyond differential equations, and may involve the approximation of functions, matrix manipulation, or finding maxima and minima. This subject is part of mathematics, rather than computer programming, because there are significant theoretical issues to solve. For example, where possible, we need to choose methods that are not too sensitive to small changes in the input data. We also need to use methods that are efficient, and give high accuracy for a reasonable amount of computing time.

In many cases, an exact solution of a differential equation cannot be found, and the most useful approach is to use a computer to find a numerical solution. The study of *numerical methods* for the solution of differential equations is a large area of knowledge. This section describes *Euler's method*, which is a simple numerical method for first-order differential equations. Other methods are often used by experts, but most of these are just refinements of Euler's method.

Let us suppose that we wish to find a solution to the first-order differential equation

$$\frac{dy}{dx} = f(x, y)$$

with initial condition $y(x_0) = y_0$ (i.e. we seek a solution that passes through the point (x_0, y_0)). We will consider the solution for $x > x_0$.

We will approximate the solution of this initial-value problem by moving in a sequence of straight-line steps. Corresponding to the given initial condition $y(x_0) = y_0$, there is a point P_0 in the xy-plane with coordinates (x_0, y_0), and this is our starting point. At P_0, the function $y(x)$ has a particular slope, namely $f(x_0, y_0)$. We move off from P_0 along a straight line that has this slope, and continue until we have travelled a *small* horizontal distance h to the right of P_0. The point that has now been reached is labelled P_1, as in Figure 12 (where the distance h is exaggerated for clarity).

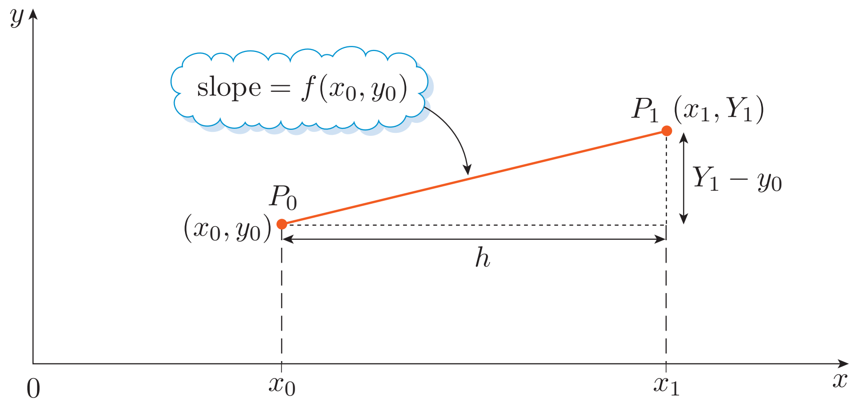


Figure 12 Graphical representation of the first step in Euler's method for numerically solving $y' = f(x, y)$

We denote the coordinates of P_1 by (x_1, Y_1). For comparison, the *exact* solution of the initial-value problem passes through a point (x_1, y_1), where $y_1 = y(x_1)$ is the value of the exact solution at $x = x_1$. Note that we cannot claim that $Y_1 = y_1$. This is unlikely to happen unless the exact solution function follows a straight line as x moves from x_0 to x_1. However, the hope is that because we headed off from x_0 along the correct slope, Y_1 will be reasonably close to the exact value, y_1. Furthermore, we expect that this approximation becomes better as the length h is decreased.

The next thing that we need to do is obtain formulas for x_1 and Y_1 in terms of the known quantities x_0, y_0, h and $f(x_0, y_0)$. Because the point P_1 is reached from P_0 by taking a step to the right of horizontal length h, we have

$$x_1 = x_0 + h. \tag{39}$$

We can also express Y_1 in terms of other quantities by equating two expressions for the slope of the line segment P_0P_1 (see Figure 12):

$$\frac{Y_1 - y_0}{h} = f(x_0, y_0),$$

which can be rearranged to give

$$Y_1 = y_0 + h\,f(x_0, y_0). \tag{40}$$

This completes the first stage of the method.

We now take a second step. This second step takes us from P_1 through a further horizontal distance h to the right, to the point labelled P_2 in Figure 13.

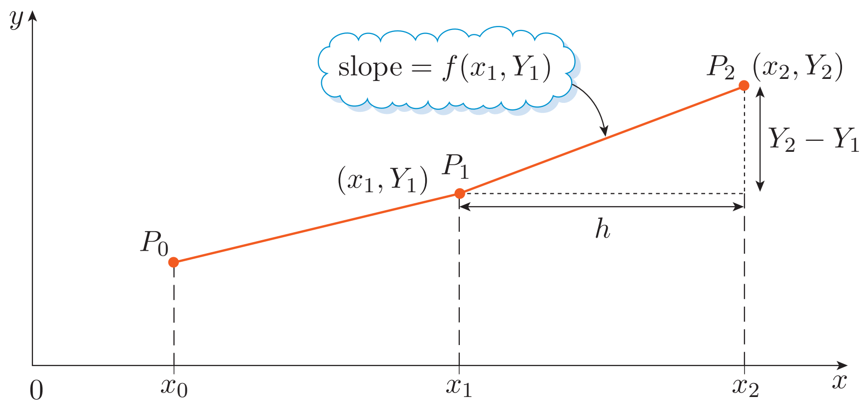


Figure 13 Graphical representation of the first two steps in Euler's method for numerically solving $y' = f(x, y)$

The coordinates of P_2 are denoted by (x_2, Y_2), and Y_2 gives an approximation to the exact solution value y_2 at the point $x = x_2$. Following the same logic as for the first step, we write

$$x_2 = x_1 + h$$

and

$$Y_2 = y_1 + h\,f(x_1, Y_1),$$

where $f(x_1, Y_1)$ is the slope dy/dx at the point $P_1 = (x_1, Y_1)$.

Having carried out two steps of the process, you can see that the same procedure can be applied to construct any number of further steps.

Suppose that after i steps we have reached the point P_i, with coordinates (x_i, Y_i). For the $(i+1)$th step, we move away from P_i along the line with slope $f(x_i, Y_i)$ (as defined by the direction field at P_i). After moving through a horizontal distance h to the right, we reach the point P_{i+1} whose coordinates are denoted by (x_{i+1}, Y_{i+1}), as illustrated in Figure 14.

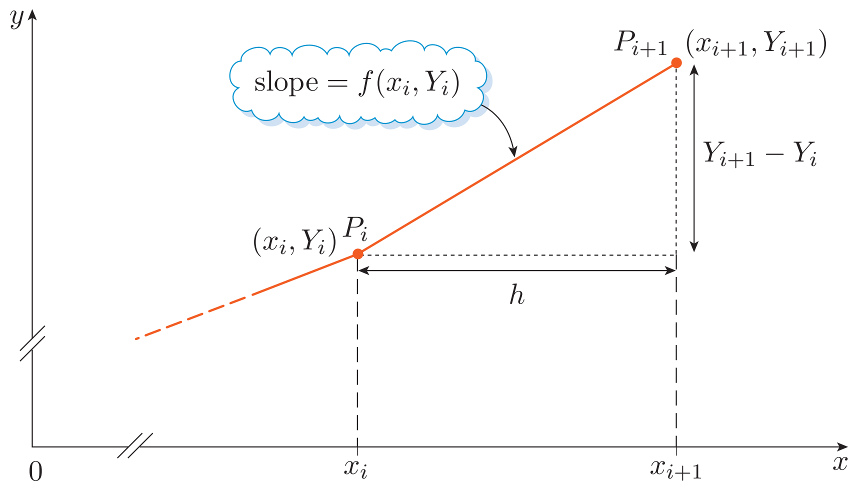


Figure 14 Graphical representation of the $(i+1)$th step of Euler's method for numerically solving $y' = f(x, y)$

The point P_{i+1} provides an approximation Y_{i+1} to the exact solution value $y_{i+1} = y(x_{i+1})$ at $x = x_{i+1}$. Arguing as before, we have

$$x_{i+1} = x_i + h \tag{41}$$

and

$$Y_{i+1} = Y_i + h\, f(x_i, Y_i). \tag{42}$$

To sum up, we have a procedure for constructing a sequence of points

$$P_i \text{ with coordinates } (x_i, Y_i) \text{ for } i = 0, 1, 2, \ldots,$$

where the values of x_i and Y_i for each value of i are determined by equations (41) and (42). The starting point for the sequence is the point P_0 with coordinates (x_0, Y_0), where $Y_0 = y_0$. The horizontal distance h by which we move to the right at each stage of the procedure is called the **step length** or **step size**.

The sequence of points $P_0 = (x_0, y_0)$, $P_1 = (x_1, Y_1)$, $P_2 = (x_2, Y_2)$, ... provides an approximate solution to the initial-value problem based on the differential equation $dy/dx = f(x, y)$ and the initial condition $y(x_0) = y_0$. In other words, when the independent variable has value x_i, the exact solution $y(x_i)$ is approximated by Y_i. The method just used to generate this sequence is called *Euler's method*.

Procedure 4 Euler's method

To apply Euler's method to the initial-value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

proceed as follows.

1. Take x_0 and $Y_0 = y_0$ as starting values, choose a step length h, and set $i = 0$.
2. Calculate the x-coordinate x_{i+1} using the formula
$$x_{i+1} = x_i + h. \qquad \text{(Eq. 41)}$$
3. Calculate a corresponding approximation Y_{i+1} to $y(x_{i+1})$, using the formula
$$Y_{i+1} = Y_i + h\, f(x_i, Y_i). \qquad \text{(Eq. 42)}$$
4. If further approximate values are required, increase i by 1 and return to Step 2.

How well does Euler's method work? Figure 15 shows the constructed sequence of points and, for comparison, shows a graph representing the exact solution of the initial-value problem. This makes it clear that the successive points $P_1, P_2, P_3, \ldots$ are only *approximations* to points on the solution curve. In fact, the situation shown in Figure 15 is typical of the behaviour of the constructed approximations, in that they gradually move further and further from the exact solution curve. This is because at each step, the direction of movement is along the slope at $P_i = (x_i, Y_i)$ and not along the slope at the position reached by the exact solution (x_i, y_i), where $y_i = y(x_i)$.

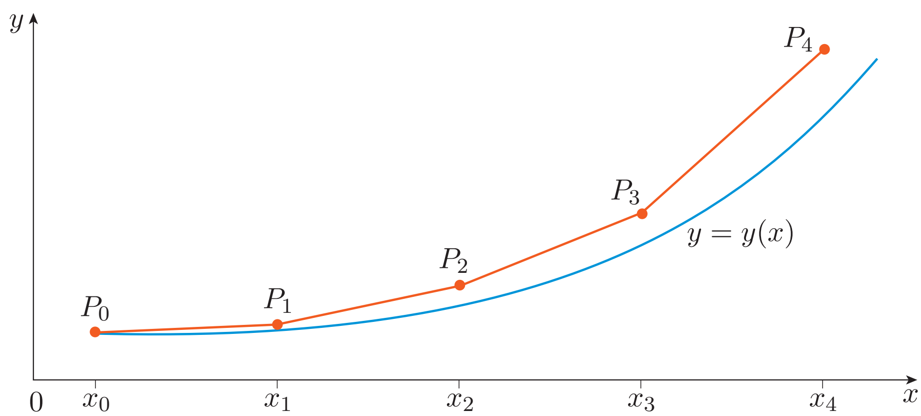


Figure 15 Graphical representation of the numerical and exact solutions of a differential equation

Improvements in accuracy can usually be achieved by *reducing the step length h*. This is illustrated in Figure 16. Of course, the improvement comes at the cost of having to use more steps, and therefore using more computer time.

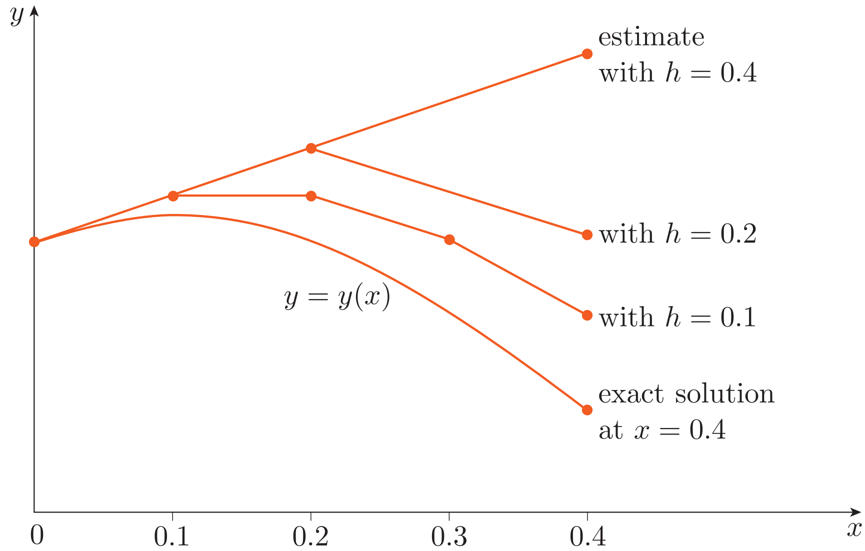


Figure 16 Comparison of the numerical and exact solutions of a differential equation for various step lengths h

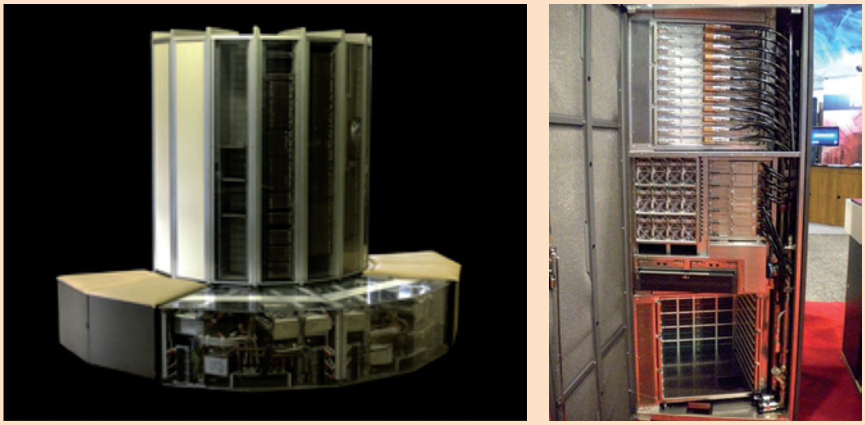
The power of computers

Euler's method is not the method of choice for solving differential equations, but it has all of the essential features of more modern methods. For large values of $x - x_0$, an accurate solution requires a very small value of h, so many steps are needed.

This feature of repeating simple calculations many times over makes Euler's method (and similar techniques) very tedious for humans, but such calculations are ideally suited to computers. In fact, this type of work was one of the first applications of computers.

Once they had taken the step of programming a computer to solve equations, the ambitions of scientists to explore complex phenomena developed rapidly. Nowadays, it is not uncommon to program a computer to solve systems of differential equations with thousands of variables. Thirty years ago this might have been done using a supercomputer, such as the Cray-1 shown in Figure 17(a). This machine was state-of-the-art in 1978 when the first models were sold for approximately 8 million dollars. Nowadays its performance is easily surpassed by a typical home computer, and supercomputers have developed even further.

Figure 17(b) shows a rack from the Japanese K supercomputer (2009), which can complete in one second computations that would have taken the original Cray-1 four years.



(a) (b)

Figure 17 (a) The Cray-1 supercomputer, introduced in 1978; (b) a rack from the Japanese K supercomputer, introduced in 2009

Learning outcomes

After studying this unit, you should be able to do the following.

- Understand and use the basic terminology relating to differential equations and their solutions.
- Check by substitution whether a given function is a solution of a given first-order differential equation or initial-value problem.
- Find from the general solution of a first-order differential equation the particular solution that satisfies a given initial condition.
- Recognise when a first-order differential equation is soluble by direct integration, and carry out that integration when appropriate.
- Recognise when a first-order differential equation is separable, and apply the method of separation of variables.
- Recognise when a first-order differential equation is linear, and solve such an equation by the integrating factor method.

Solutions to exercises

Solution to Exercise 1

We have $r(P) = k\left(1 - \dfrac{P}{M}\right)$, so we simply need to solve the following pair of simultaneous equations:

$$k\left(1 - \frac{10}{M}\right) = 1,$$

$$k\left(1 - \frac{10\,000}{M}\right) = 0.$$

From the second equation, since $k > 0$, we see immediately that $M = 10\,000$. Substituting in the first equation leads to

$$k\,\frac{999}{1000} = 1, \quad \text{so} \quad k = \frac{1000}{999}.$$

Solution to Exercise 2

(a) In each case, differences in notation notwithstanding, the differential equation has the form

$$\frac{dy}{dx} = f(x,y),$$

and we need to show that a given function $y = y(x)$ satisfies this equation. We can do this by substituting $y = y(x)$ into both sides of the equation and showing that the results are identical. Alternatively, if the right-hand side is simple, we can substitute $y = y(x)$ into the left-hand side and then rearrange the result to show that it is equal to the right-hand side.

In this case, to show that $y = 2e^x - (x^2 + 2x + 2)$ satisfies

$$\frac{dy}{dx} = f(x,y) = y + x^2,$$

we first differentiate y to get

$$\frac{dy}{dx} = 2e^x - 2x - 2.$$

Then, substituting the expression for y into $f(x,y)$, we get

$$f(x,y) = y + x^2 = 2e^x - (x^2 + 2x + 2) + x^2 = 2e^x - 2x - 2,$$

as required.

(b) To show that $y = \frac{1}{2}x^2 + \frac{3}{2}$ satisfies

$$\frac{dy}{dx} = x,$$

we simply differentiate y to get

$$\frac{dy}{dx} = x,$$

which establishes the required result.

(c) To show that $u = 2e^{x^2/2}$ satisfies

$$u' = f(x,u) = xu,$$

we differentiate u to get

$$u' = \frac{du}{dx} = 2xe^{x^2/2},$$

and then note that the right-hand side of the differential equation is

$$f(x,u) = xu = 2xe^{x^2/2},$$

as required.

(d) To show that

$$y = \sqrt{\frac{27 - x^2}{3}} = \left(\frac{27 - x^2}{3}\right)^{1/2},$$

with $(-3\sqrt{3} < x < 3\sqrt{3})$, satisfies

$$\frac{dy}{dx} = f(x,y) = -\frac{x}{3y} \quad (y \neq 0),$$

we first differentiate y to get

$$\frac{dy}{dx} = -\frac{x}{3}\left(\frac{27 - x^2}{3}\right)^{-1/2}.$$

Then, substituting the expression for y into $f(x,y)$, we get

$$f(x,y) = -\frac{x}{3y} = -\frac{x}{3}\left(\frac{27 - x^2}{3}\right)^{-1/2},$$

as required.

(e) To show that $y = t + e^{-t}$ satisfies

$$\dot{y} = f(t,y) = -y + t + 1,$$

we first differentiate y to get

$$\dot{y} = \frac{dy}{dt} = 1 - e^{-t}.$$

Then, substituting the expression for y into $f(t,y)$, we get

$$f(t,y) = -y + t + 1 = -(t + e^{-t}) + t + 1 = 1 - e^{-t},$$

as required.

(f) To show that $y = t + Ce^{-t}$ satisfies

$$\dot{y} = f(t, y) = -y + t + 1,$$

we first differentiate y to get

$$\dot{y} = \frac{dy}{dt} = 1 - Ce^{-t}.$$

Then, substituting the expression for y into $f(t, y)$, we get

$$f(t, y) = -y + t + 1 = -(t + Ce^{-t}) + t + 1 = 1 - Ce^{-t},$$

as required.

Solution to Exercise 3

(a) To verify that $y = C - \frac{1}{3}e^{-3x}$ satisfies

$$\frac{dy}{dx} = e^{-3x},$$

we differentiate y to get

$$\frac{dy}{dx} = -\tfrac{1}{3}(-3e^{-3x}) = e^{-3x},$$

as required.

(b) To verify that $u = Ce^t - t - 1$ satisfies

$$\dot{u} = f(t, u) = t + u,$$

we differentiate u to get

$$\dot{u} = \frac{du}{dt} = Ce^t - 1.$$

Then, substituting the expression for u into $f(t, u)$, we get

$$f(t, u) = t + u = Ce^t - 1,$$

as required.

(c) To verify that $P(t) = CMe^{kt}/(1 + Ce^{kt})$ satisfies

$$\frac{dP}{dt} = f(t, P) = kP\left(1 - \frac{P}{M}\right),$$

we differentiate $P(t)$ using the quotient rule. This gives

$$\frac{dP}{dt} = \frac{(CMke^{kt})(1 + Ce^{kt}) - (CMe^{kt})(Cke^{kt})}{(1 + Ce^{kt})^2} = \frac{CMke^{kt}}{(1 + Ce^{kt})^2}.$$

Then, substituting the expression for P into $f(t, P)$, we get

$$f(t, P) = k\,\frac{CMe^{kt}}{1 + Ce^{kt}}\left(1 - \frac{Ce^{kt}}{1 + Ce^{kt}}\right) = \frac{CMke^{kt}}{(1 + Ce^{kt})^2},$$

as required.

Solution to Exercise 4

(a) From Exercise 3(c) we know that

$$P(t) = \frac{CMe^{kt}}{1 + Ce^{kt}} = \frac{10Ce^{0.15t}}{1 + Ce^{0.15t}}$$

is a solution of the differential equation. Because $e^0 = 1$, the initial condition $P(0) = 1$ then implies that

$$1 = \frac{10C}{1 + C}, \quad \text{so} \quad C = \tfrac{1}{9}.$$

The particular solution consistent with the initial condition is therefore

$$P(t) = \frac{\frac{10}{9}e^{0.15t}}{1 + \frac{1}{9}e^{0.15t}} = \frac{10e^{0.15t}}{9 + e^{0.15t}}.$$

(b) Dividing top and bottom by $e^{0.15t}$, we see that

$$P(t) = \frac{10}{9e^{-0.15t} + 1}.$$

For large values of t, the exponential term in the denominator will be very small. The result is that P will approach the value 10 in the long term. As P is the population size measured in hundreds of thousands, the population is predicted to approach one million in the long term.

Solution to Exercise 5

(a) We apply direct integration to find the general solution. In each case, C is an arbitrary constant.

The differential equation $dy/dx = 6x$ has general solution

$$y = \int 6x\,dx = 3x^2 + C.$$

From the initial condition $y(1) = 5$, we have $5 = 3 + C$, so $C = 2$. The solution to the initial-value problem is therefore

$$y = 3x^2 + 2.$$

(b) The differential equation $dv/du = e^{4u}$ has general solution

$$v = \int e^{4u}\,du = \tfrac{1}{4}e^{4u} + C.$$

From the initial condition $v(0) = 2$, we have $2 = \frac{1}{4} + C$, so $C = \frac{7}{4}$. The solution to the initial-value problem is therefore

$$v = \tfrac{1}{4}(e^{4u} + 7).$$

(c) The differential equation $\dot{y} = 5\sin 2t$ has general solution

$$y = \int 5\sin 2t\,dt = -\tfrac{5}{2}\cos 2t + C.$$

From the initial condition $y(0) = 0$, we have $0 = -\frac{5}{2} + C$, so $C = \frac{5}{2}$. The solution to the initial-value problem is therefore

$$y = \tfrac{5}{2}(1 - \cos 2t).$$

Solution to Exercise 6

(a) The differential equation $dy/dx = xe^{-2x}$ has general solution

$$y = \int xe^{-2x}\,dx.$$

The integral can be found using integration by parts (see Unit 1).

Since differentiating x simplifies it, we take $f(x) = x$ and $g'(x) = e^{-2x}$. Then $f'(x) = 1$ and $g(x) = -\frac{1}{2}e^{-2x}$. So, using the formula

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx,$$

we get

$$\begin{aligned}\int xe^{-2x}\,dx &= -\tfrac{1}{2}xe^{-2x} + \int \tfrac{1}{2}e^{-2x}\,dx \\ &= -\tfrac{1}{2}xe^{-2x} - \tfrac{1}{4}e^{-2x} + C,\end{aligned}$$

where C is an arbitrary constant. The general solution of the differential equation is therefore

$$y = -\tfrac{1}{4}(2x+1)e^{-2x} + C.$$

(b) The differential equation $\dot{p} = t/(1+t^2)$ has general solution

$$p = \int \frac{t}{1+t^2}\,dt.$$

Using the hint provided, we make the substitution $u = 1+t^2$, for which $du/dt = 2t$. Writing the required integral as

$$\int \frac{t}{1+t^2}\,dt = \tfrac{1}{2}\int \frac{2t}{1+t^2}\,dt,$$

we then obtain

$$\begin{aligned}\int \frac{t}{1+t^2}\,dt = \int \frac{1}{u}\,\frac{du}{dt}\,dt &= \tfrac{1}{2}\int \frac{1}{u}\,du \\ &= \tfrac{1}{2}\ln u + C \quad (\text{since } u = 1+t^2 > 0) \\ &= \tfrac{1}{2}\ln(1+t^2) + C,\end{aligned}$$

where C is an arbitrary constant. The general solution of the differential equation is therefore

$$p = \tfrac{1}{2}\ln(1+t^2) + C.$$

Solution to Exercise 7

(a) The differential equation is

$$\frac{dy}{dx} = \frac{y-1}{x}, \quad \text{where } x > 0,$$

which is of the form $dy/dx = g(x)\,h(y)$ with $g(x) = 1/x$ and $h(y) = y - 1$. For $h(y) = y - 1 \neq 0$, Procedure 2 gives

$$\int \frac{1}{y-1}\,dy = \int \frac{1}{x}\,dx.$$

Since $x > 0$ and $y - 1 \neq 0$, integration produces

$$\ln|y-1| = \ln x + C,$$

where C is an arbitrary constant. Taking the exponential of both sides of this equation, we get

$$|y-1| = e^{\ln x + C} = e^C e^{\ln x} = e^C x.$$

Hence

$$y = 1 \pm e^C x.$$

We therefore have a family of solutions given by

$$y(x) = 1 + Bx,$$

where $B = \pm e^C$ is an arbitrary non-zero constant.

The special case of the constant function $y(x) = 1$ is also a solution since if we substitute this into both sides of the differential equation, we get zero on both sides. This special case can be incorporated into the main family of solutions by setting $B = 0$.

So we conclude that any function of the form

$$y(x) = 1 + Bx$$

is a solution of the differential equation for $x > 0$. In fact, it is the general solution of the equation. (In Section 3, Example 7(a), you will see this established by a different method.)

(b) The differential equation is

$$\frac{dy}{dx} = \frac{2y}{1+x^2},$$

which is of the form $dy/dx = g(x)\,h(y)$ with $g(x) = 1/(1+x^2)$ and $h(y) = 2y$. For $h(y) = 2y \neq 0$, Procedure 2 gives

$$\int \frac{1}{y}\,dy = \int \frac{2}{1+x^2}\,dx.$$

For $y \neq 0$, integration produces

$$\ln|y| = 2(\arctan x + C),$$

where C is an arbitrary constant.

On solving this equation for y, we obtain

$$y = \pm e^{2\arctan x + 2C} = \pm e^{2C} e^{2\arctan x} = Be^{2\arctan x},$$

where $B = \pm e^{2C}$ is a non-zero but otherwise arbitrary constant.

The special case of the constant function $y(x) = 0$ is also a solution, since if we substitute this into both sides of the differential equation, we get zero on both sides. This special case can be incorporated into the main family of solutions by setting $B = 0$.

So we conclude that any function of the form

$$y(x) = Be^{2\arctan x},$$

where B is an arbitrary constant, is a solution of the differential equation. In fact, it is the general solution of the equation. (In Section 3, Example 7(b), you will see this established by a different method.)

Solution to Exercise 8

The differential equation is

$$\frac{dv}{du} = e^{u+v} = e^u e^v.$$

Dividing through by e^v and integrating with respect to u, we obtain

Note that $e^v > 0$.

$$\int e^{-v}\, dv = \int e^u\, du.$$

Integration produces

$$-e^{-v} = e^u + C,$$

where C is an arbitrary constant. So

$$e^{-v} = -e^u - C.$$

Taking the logarithm of both sides, we get

$$-v = \ln(-e^u - C),$$

so

$$v = -\ln(B - e^u),$$

where $B = -C$.

Since the argument of the ln function must be positive, we require that $B - e^u > 0$, so $B > e^u$. Hence B must be positive. Taking the logarithm of both sides gives $\ln B > \ln e^u$, so $\ln B > u$.

Therefore the general solution is

$$v = -\ln(B - e^u) \quad (u < \ln B),$$

where B is an arbitrary positive constant.

The initial condition $v(0) = 0$ gives $0 = -\ln(B - e^0)$, so $B - e^0 = 1$ and hence $B = 2$. The solution to the initial-value problem is therefore

$$v = -\ln(2 - e^u) \quad (u < \ln 2).$$

Solution to Exercise 9

(a) Each of these differential equations can be solved by separation of variables.

In this case the differential equation is

$$u' = \frac{du}{dx} = xu.$$

For the cases where $u \neq 0$, we divide through by u and integrate with respect to x. This gives

$$\int \frac{1}{u}\,du = \int x\,dx.$$

Integration produces

$$\ln|u| = \tfrac{1}{2}x^2 + C,$$

where C is an arbitrary constant. On solving this equation for u, we obtain

$$u = \pm e^{x^2/2+C} = \pm e^C e^{x^2/2} = Be^{x^2/2},$$

where $B = \pm e^C$ is a non-zero but otherwise arbitrary constant.

The special case of the constant function $u(x) = 0$ is also a solution, since if we substitute this into both sides of the differential equation, we get zero on both sides. This special case can be incorporated into the main family of solutions by setting $B = 0$.

So we conclude that any function of the form

$$u(x) = Be^{x^2/2},$$

where B is an arbitrary constant, is a solution of the differential equation. This is the general solution.

(You verified that $u = 2e^{x^2/2}$ is a particular solution of this differential equation in Exercise 2(c).)

(b) The differential equation is

$$\dot{x} = \frac{dx}{dt} = 1 + x^2.$$

We divide through by $1 + x^2$ and integrate with respect to t. This gives

$$\int \frac{1}{1+x^2}\,dx = \int 1\,dt.$$

Integration then produces

$$\arctan x = t + C,$$

where C is an arbitrary constant. On solving for x, we get the general solution

$$x(t) = \tan(t + C).$$

We must restrict the domain of $x(t)$ to avoid values of t where tan is undefined. That is, we must ensure that $t + C \neq \left(n + \tfrac{1}{2}\right)\pi$, where n is

an integer, but the precise choice of domain will depend on the initial conditions.

Solution to Exercise 10

(a) The given equation is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

First, note that the constant functions $P(t) = 0$ and $P(t) = M$ are both solutions of the differential equation, giving zero on both sides. The function $P(t) = 0$ is not allowed, however, since we are told that $P(t) > 0$. Ignoring for the moment the possibility that $P(t) = M$, we can use the method of separation of variables to obtain

$$\int \frac{1}{P(1 - P/M)}\, dP = \int k\, dt.$$

The integral on the left-hand side can be evaluated using the result given in the question with $a = 1/M$. We get

$$-\ln\left|\frac{1}{P} - \frac{1}{M}\right| = kt + C,$$

where C is an arbitrary constant. Hence

$$\left|\frac{1}{P} - \frac{1}{M}\right| = e^{-kt-C} = e^{-C}e^{-kt},$$

thus

$$\frac{1}{P} = \frac{1}{M} \pm e^{-C}e^{-kt} = \frac{1}{M} + Be^{-kt},$$

where B is a non-zero but otherwise arbitrary constant.

Now consider the constant solution $P(t) = M$. This can be incorporated into the above family of solutions by taking $B = 0$. So the restriction $B \neq 0$ can be dropped and the general solution of the differential equation is

$$P(t) = \left(\frac{1}{M} + Be^{-kt}\right)^{-1},$$

where B is an arbitrary constant.

From the initial condition $P(0) = 2M$, we get

$$\frac{1}{2M} = \frac{1}{M} + Be^{0}, \quad \text{so} \quad B = -\frac{1}{2M}.$$

The solution to the initial-value problem is therefore

$$\begin{aligned} P(t) &= \left(\frac{1}{M} - \frac{1}{2M}e^{-kt}\right)^{-1} \\ &= \frac{2M}{2 - e^{-kt}}. \end{aligned}$$

(b) As $t \to \infty$ we have $e^{-kt} \to 0$, so the value of $P(t)$ approaches M.

Solution to Exercise 11

(a) The equation $dy/dx + x^3y = x^5$ is linear, with $g(x) = x^3$ and $h(x) = x^5$.

(b) The equation $dy/dx = x \sin x$ is linear, with $g(x) = 0$ (for all x) and $h(x) = x \sin x$.

(c) The equation $dz/dt = -3z^{1/2}$ is not linear (because of the $z^{1/2}$ term).

(d) The equation $\dot{y} + y^2 = t$ is not linear (because of the y^2 term).

(e) The equation $x\,(dy/dx) + y = y^2$ is not linear (because of the y^2 term).

(f) The equation $(1 + x^2)\,(dy/dx) + 2xy = 3x^2$ is linear, since we can divide through by $1 + x^2$ to obtain

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{3x^2}{1+x^2},$$

which is of linear form with $g(x) = 2x/(1 + x^2)$ and $h(x) = 3x^2/(1 + x^2)$.

Solution to Exercise 12

The differential equation

$$\frac{dy}{dx} + y = e^{2x}$$

has the same form as equation (26), with $A = 1$ and $h(x) = e^{2x}$. Multiplying both sides of the differential equation by the factor $e^{Ax} = e^x$, we obtain

$$e^x \frac{dy}{dx} + e^x y = e^x e^{2x} = e^{3x},$$

and this can be written in the form

$$\frac{d}{dx}(e^x y) = \int e^{3x}\, dx.$$

Integrating both sides then gives

$$e^x y = \tfrac{1}{3}e^{3x} + C,$$

so

$$y(x) = \tfrac{1}{3}e^{2x} + Ce^{-x}.$$

The initial condition $y(0) = 0$ gives $0 = \frac{1}{3} + C$, so $C = -\frac{1}{3}$. The particular solution consistent with the given initial condition is therefore

$$y(x) = \tfrac{1}{3}(e^{2x} - e^{-x}).$$

We check this by differentiating $y(x)$ to get

$$\frac{dy}{dx} = \tfrac{1}{3}(2e^{2x} + e^{-x}),$$

so

$$\frac{dy}{dx} + y = \tfrac{1}{3}(2e^{2x} + e^{-x}) + \tfrac{1}{3}(e^{2x} - e^{-x}) = e^{2x},$$

as required.

Solution to Exercise 13

(a) The given equation is

$$\frac{dy}{dx} - y = e^x \sin x.$$

Comparison with equations (25) and (32) shows that the integrating factor is

$$p(x) = \exp\left(\int (-1)\,dx\right) = \exp(-x) = e^{-x}.$$

Multiplying through by $p(x)$ then gives

$$e^{-x}\frac{dy}{dx} - e^{-x}y = \sin x.$$

Thus the differential equation can be rewritten as

$$\frac{d}{dx}(e^{-x}y) = \sin x.$$

(You should check that the derivative on the left-hand side is equal to the left-hand side of the preceding equation.)

On integrating, we find the general solution

$$e^{-x}y = -\cos x + C,$$

or equivalently,

$$y(x) = e^x(C - \cos x),$$

where C is an arbitrary constant.

(b) The given equation, when rearranged into the form of equation (25), is

$$\frac{dy}{dx} - y = x.$$

This has the same left-hand side as the differential equation in part (a), and hence the same integrating factor, $p(x) = e^{-x}$. Multiplying through by $p(x)$ gives

$$e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{-x}.$$

Thus the differential equation can be rewritten as

$$\frac{d}{dx}(e^{-x}y) = xe^{-x}.$$

Integrating both sides, we get

$$e^{-x}y = \int xe^{-x}\,dx.$$

The integral on the right-hand side is integrated by parts, using the formula

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

with $f(x) = x$ and $g'(x) = e^{-x}$.

We have $f'(x) = 1$ and $g(x) = -e^{-x}$, so

$$\begin{aligned} e^{-x}y &= -xe^{-x} + \int e^{-x}\,dx \\ &= -xe^{-x} - e^{-x} + C \\ &= C - (x+1)e^{-x}, \end{aligned}$$

where C is an arbitrary constant. Multiplying through by e^x, the explicit form of the general solution is

$$y = Ce^x - (x+1).$$

Solution to Exercise 14

(a) The given equation, when rearranged into the form of equation (25), is

$$\frac{du}{dx} - xu = 0.$$

The integrating factor is

$$p(x) = \exp\left(\int (-x)\,dx\right) = \exp(-x^2/2) = e^{-x^2/2}.$$

Multiplying through by $p(x)$ gives

$$e^{-x^2/2}\frac{du}{dx} - xe^{-x^2/2}u = 0.$$

Thus the differential equation can be rewritten as

$$\frac{d}{dx}(e^{-x^2/2}u) = 0.$$

(You should check that the derivative on the left-hand side is equal to the left-hand side of the preceding equation.)

On integrating, we find the general solution

$$e^{-x^2/2}u = C,$$

or equivalently,

$$u(x) = Ce^{x^2/2},$$

where C is an arbitrary constant.

From the initial condition $u(0) = 2$, we have $2 = Ce^0$, so $C = 2$. Hence the solution of the initial-value problem is

$$u(x) = 2e^{x^2/2}.$$

(b) After division by t (which is allowed because we are told that $t > 0$), the given equation can be written as

$$\frac{dy}{dt} + \frac{2}{t}y = t.$$

The integrating factor is

$$p(x) = \exp\left(\int \frac{2}{t}\,dt\right) = \exp(2\ln t) = \exp(\ln(t^2)) = t^2.$$

Multiplying through by $p(t)$ gives

$$t^2 \frac{dy}{dt} + 2ty = t^3.$$

Thus the differential equation can be rewritten as

$$\frac{d}{dt}(t^2 y) = t^3.$$

(Again, it's worth checking that the derivative on the left-hand side gives the left-hand side of the preceding equation.)

On integrating, we find the general solution

$$t^2 y = \tfrac{1}{4}t^4 + C,$$

or equivalently,

$$y(t) = \tfrac{1}{4}t^2 + Ct^{-2},$$

where C is an arbitrary constant.

From the initial condition $y(1) = 1$, we have $1 = \frac{1}{4} + C$, so $C = \frac{3}{4}$. Hence the solution of the initial-value problem is

$$y(t) = \tfrac{1}{4}(t^2 + 3t^{-2}).$$

Solution to Exercise 15

(a) The given equation is

$$\frac{dy}{dt} + y = t + 1.$$

So the integrating factor is

$$p(t) = \exp\left(\int 1\,dt\right) = \exp(t) = e^t.$$

Multiplying through by $p(t)$ gives

$$e^t \frac{dy}{dt} + e^t y = (t+1)e^t.$$

Thus the differential equation can be rewritten as

$$\frac{d}{dt}(e^t y) = (t+1)e^t.$$

So

$$e^t y = \int (t+1)e^t\,dt.$$

The integral on the right-hand side is evaluated by parts, using $f(t) = t + 1$ and $g'(t) = e^t$. We have $f'(t) = 1$ and $g(t) = e^t$, so

$$\begin{aligned} e^t y &= (t+1)e^t - \int e^t\,dt \\ &= (t+1)e^t - e^t + C \\ &= te^t + C, \end{aligned}$$

where C is an arbitrary constant.

Multiplying through by e^{-t}, the general solution in explicit form is

$$y(t) = Ce^{-t} + t.$$

From the initial condition $y(1) = 0$, we have $0 = Ce^{-1} + 1$, so $C = -e$. Hence the solution of the initial-value problem is

$$y(t) = t - e^{1-t}.$$

(b) After division by e^{3t} and rearrangement, the given equation becomes $dy/dt + y = e^{-3t}$. This has the same left-hand side as the differential equation in part (a), and hence the same integrating factor, $p(t) = e^t$. Multiplying through by $p(t)$ gives

$$e^t \frac{dy}{dt} + e^t y = e^{-2t}.$$

Thus the differential equation can be rewritten as

$$\frac{d}{dt}(e^t y) = e^{-2t}.$$

On integrating, we find the general solution

$$e^t y = -\tfrac{1}{2}e^{-2t} + C,$$

or equivalently,

$$y(t) = Ce^{-t} - \tfrac{1}{2}e^{-3t},$$

where C is an arbitrary constant.

From the initial condition $y(0) = 3$, we have $3 = Ce^0 - \frac{1}{2}e^0$, so $C = \frac{7}{2}$. Hence the solution of the initial-value problem is

$$y(t) = \tfrac{1}{2}(7e^{-t} - e^{-3t}).$$

Solution to Exercise 16

(a) After division by x (where $x > 0$), the given equation becomes $dy/dx - (3/x)y = 1$. The integrating factor is therefore

$$\begin{aligned} p(x) &= \exp\left(\int \left(-\frac{3}{x}\right) dx\right) \\ &= \exp(-3\ln x) \quad (\text{since } x > 0) \\ &= \exp(\ln(x^{-3})) \\ &= x^{-3}. \end{aligned}$$

Multiplying through by $p(x)$ gives

$$x^{-3}\frac{dy}{dx} - 3x^{-4}y = x^{-3}.$$

Thus the differential equation can be rewritten as

$$\frac{d}{dx}(x^{-3}y) = x^{-3}.$$

On integrating, we find the general solution

$$x^{-3}y = -\tfrac{1}{2}x^{-2} + C,$$

or equivalently,

$$y(x) = Cx^3 - \tfrac{1}{2}x,$$

where C is an arbitrary constant.

(b) The given equation is $dv/dt + 4v = 3\cos 2t$. The integrating factor is

$$p(t) = \exp\left(\int 4\,dt\right) = \exp(4t) = e^{4t}.$$

Multiplying through by $p(t)$ gives

$$e^{4t}\frac{dv}{dt} + 4e^{4t}v = 3e^{4t}\cos 2t.$$

Thus the differential equation can be rewritten as

$$\frac{d}{dt}(e^{4t}v) = 3e^{4t}\cos 2t.$$

On integrating (using the hint for the right-hand side, with $a = 4$ and $b = 2$), we find

$$e^{4t}v = \tfrac{3}{20}e^{4t}(4\cos 2t + 2\sin 2t) + C,$$

where C is an arbitrary constant. Multiplying through by e^{-4t}, the general solution in explicit form is

$$v(t) = \tfrac{3}{10}(2\cos 2t + \sin 2t) + Ce^{-4t}.$$

Solution to Exercise 17

(a) and (d) require the integrating factor method.

(b) is best solved by direct integration.

(c) can be solved by separation of variables or the integrating factor method.

(e) requires separation of variables.

Solution to Exercise 18

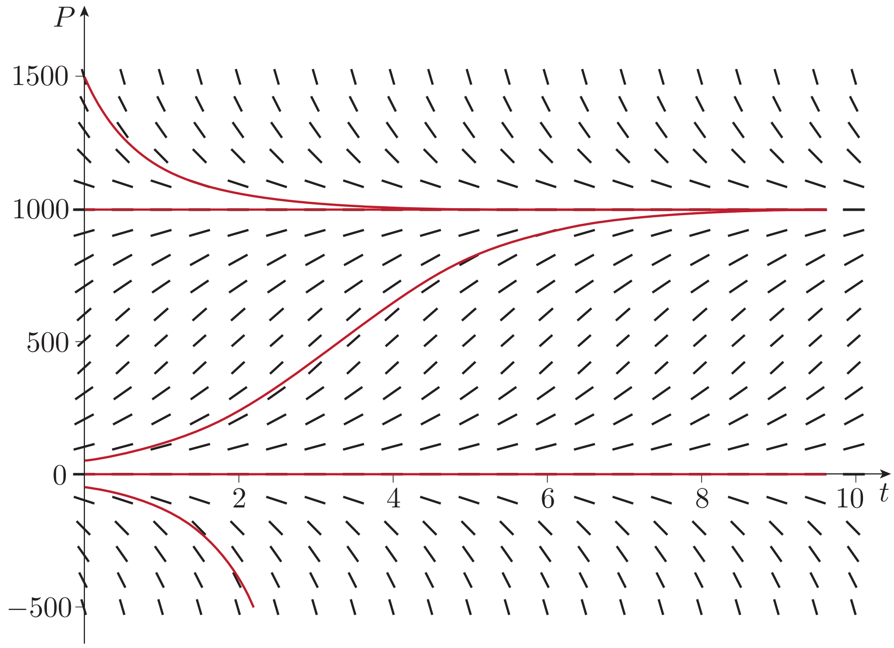
(a) The diagram shows that the slope is zero at all points on the horizontal lines $P = 0$ and $P = 1000$, so these correspond to constant solutions of the differential equation. (As pointed out earlier in the text, these two solutions can also be spotted directly from the form of the differential equation.)

The graphs of solutions through a starting point above the line $P = 1000$ appear to decrease, but at a slower and slower rate, tending from above towards the asymptote $P = 1000$ as t increases.

The graphs of solutions through starting points in the region $0 < P < 1000$ are increasing, with slope growing before the level $P = 500$ is reached and declining thereafter. For large values of t, these graphs tend from below towards the asymptote $P = 1000$.

For a starting point in the region $P < 0$, the graphs decrease without limit and with steeper and steeper slope.

These various cases are illustrated in the figure below.



(b) If the differential equation is considered as a model of population behaviour, then the region $P < 0$ must be excluded. The above analysis leads to the following predictions for the population.

- If the population size is zero at the start, then it remains zero.
- If the population size is 1000 at the start, then it remains fixed at this level.
- If the population starts at a level higher than 1000, then it declines (more and more gradually) towards 1000.
- If the population starts at a level below 1000 (but above 0), then it increases (more and more gradually) towards 1000.
- Solutions with $P < 0$ are unphysical in the context of population models.

Acknowledgements

Grateful acknowledgement is made to the following sources:

Figure 3: Andrew Butko.

Figure 4: Wallace 63 / http://commons.wikimedia.org/wiki/File:Smilodon_head.jpg. This file is licensed under the Creative Commons Attribution-Share Alike 3.0.

Figure 7: NASA.

Figure 8: NASA.

Figure 17(a): Taken from http://en.wikipedia.org/wiki/File:Cray_1_IMG_9126.jpg. This file is licensed under the Creative Commons Attribution-Share Alike 2.0 France licence.

Figure 17(b): Taken from http://en.wikipedia.org/wiki/File:Keisoku-Fujitsu.jpg. This file is licensed under the Creative Commons Attribution-Share Alike 3.0 Unported licence.

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